Equations of 2-body motion

The fundamental eqn. of classical astrodynamics is Newton’s Law of Universal Gravitation:

\[ \vec{F}_g = -G \sum_i \frac{M_i}{r_i^2} \hat{r}_i \]  

(1)

We are interested in satellites in orbits about single planets, so (1) reduces to the ‘2-body’ form:

\[ \vec{F}_s = -G \frac{Mm_s}{r^2} \hat{r}_s \]  

(2)

where \( F_s \) is the gravitational force on the satellite and \( r_s \) is its position vector relative to the center of the planet.

We can use Newton’s 2nd Law to get rid of \( m_s \):

\[ \vec{F}_s = m_s \ddot{r}_s = -G \frac{Mm_s}{r^2} \hat{r} \]

\[ \ddot{r}_s + G \frac{M}{r^3} \hat{r} = 0 \]  

(3)

This eqn can be solved, leading to the eqn of the satellite orbit:

...
\[ r = \frac{a(1 - e^2)}{1 + e \cdot \cos \theta} \]  

(4)

and to Kepler’s 2nd Law:

\[ \frac{d}{dt} (r^2 \dot{\theta}) = 0 \]  

(5)

Eqn (4) says the satellite orbit is a conic section with the planet at one focus; we’ll see that the type of conic section depends on the eccentricity, \( e \), of the orbit. An important consequence of this is that the orbit, planet and satellite-center are confined to a plane (the ‘orbital plane’) for 2-body motion.

The quantity \( r^2 \dot{\theta} \) is the angular momentum, so (5) is a statement of the conservation of angular momentum.

\[ r^2 \dot{\theta} = \text{constant} \equiv h \]  

(6)

**Energy-**

Gravity is a conservative field, therefore the total energy of the satellite:
\[
KE + PE = \frac{1}{2} m_s v^2 - G \frac{M m_s}{r} = \text{constant}
\]

or:
\[
E \equiv \frac{1}{2} v^2 - G \frac{M}{r}
\]  \(\text{(7)}\)

Notice that the PE part of the energy, as defined here, is zero for at “infinity”, and negative for anything less. Also, E is really the energy per unit mass. Thus:

- only objects that reach \(\infty\) with some excess velocity (ie. that \textit{escape the planet}) have positive values for E.

- objects that can barely reach \(\infty\) (ie. whose \(v \to 0\) as \(r \to \infty\)) have \(E=0\).

- objects in bound (ie. closed) orbits have \(E < 0\).

Now we’re ready to look at orbit shapes.

\textbf{Orbits and energy-}

The negative-energy, closed orbits obviously have to be ellipses. These correspond to values of
e between 0 and 1. The case of $e=0$ reduces eqn (4) to $r = a$ ($a$ being the semi-major axis); the special case of a circular orbit.

The other special case, $e=1$ corresponds to an “ellipse with an infinite semi-major axis” (try it and see) a.k.a., a parabola.

The parabola represents a critical case; $e>0$ orbits correspond to hyperbolic orbits, which in spacecraft terms are ‘escape orbits’.

**Circular orbits**-

Notice that, according to (5), both the radius (thus altitude) and the angular velocity of a satellite in an elliptical orbit vary with time; satellites move slower the more distant they are from their primary and (5) governs how. This is very bad for imaging!
The red dots are positions of a satellite orbitting the origin, at fixed time intervals. The fact that the yellow triangle all have equal areas is a consequence of (5).

**BUT**, (5) *also* says that if \( r \) is constant, so is the angular velocity. This is a big reason why circular orbits are preferred for almost all planetary remote sensing applications.

**Physics 101 fact**: The acceleration of an object moving in a circle at constant angular velocity is:

\[
\ddot{r}_s = - \frac{V_{\text{circ}}^2}{r} \hat{r}
\]

But according to (3):

\[
\ddot{r}_s = - G \frac{M}{r^2} \hat{r}
\]
Thus:

\[
v_{\text{circ}} = \sqrt{\frac{GM}{r}}
\]  \( (8) \)

This equation gives the speed of \textit{any} object in circular orbit at radius \( r \) about a primary \( M \). Here are some examples:

\( (1) \) EOS-1 “Terra” is in low circular orbit (LEO) at an altitude of \(~700\) km or \( r \sim 7080\) km. It’s speed relative to Earth’s center at this altitude is \( 7.5\) km/s. The circumference of this orbit is \( 44,500\) km, so Terra’s period \( T \) is \( 44,500/7.5 = 5933 \) s (1 hr 39 min).

\( (2) \) Geosynchronous satellites orbit Earth in a circle once each day. Solving:

\[
2\pi r_{\text{geo}} = v_{\text{geo}} T \quad \text{and} \quad v_{\text{geo}} = \sqrt{\frac{GM}{r_{\text{geo}}}}
\]

simultaneously, and plugging in \( T = 24\) hrs one finds that Earth GEO is at an altitude of \( 36,000 \) km \( (r_{\text{geo}} = 42,400\) km) and \( v_{\text{geo}} = 3.1\) km/s (notice how much smaller this is than the velocity of Terra, in LEO). When placed over the equator, geosynch are stationary over one point on Earth’s surface, so they are “geostationary”. There are tons of remote sensing (e.g. weather) and comm sats in these orbits.

**Orbital elements**

In addition to an origin, 6 numbers are needed to specify an orbit. In the Cartesian case, we would need 3 position components and 3 velocity components.

An alternative, and much more intuitive set are called the orbital elements. We already have 2 of these: Semi-major axis and eccentricity describe the \textit{shape} of the elliptical orbit. But we still need to specify it's orientation in space:
Three angles take care of this: the "longitude of the ascending node" ($\Omega$), the "inclination" ($i$) and the "argument of periapsis" ($\omega$).

Note that:

- *perfectly* circular orbits would have no apsides, therefore $\omega$ would be undefined; however *real* orbits are never perfect ...
- from example (2) above that planeto-stationary satellites must have inclinations of zero (therefore $\Omega$ and $\omega$ are undefined).

- the highest latitude a satellite reaches is equal to it's inclination; for example, polar orbitting satellites . . .

(3) The vast majority of remote sensing satellites are put in near-polar LEOs*, for a number of reasons:

a. Highly inclined satellites can cover the arctic/antarctic.

b. They can observe mid- and low-latitudes several times a day, with coverage of most areas every 2-10 days, depending on swath-width, latitude, orbit details....

c. They can be orbitted so that they pass over the day side of the planet at the same time every day.....

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* a LEO is loosely defined as an orbit with apogee less than ~2000 km in altitude.
This is because real planets are oblate. This shape causes the orbits planes of inclined satellites (including planets!) to precess:

$$\Delta \Omega = 3\pi \frac{J_2 R_E^2 \cos(i)}{a^2 (1 - e^2)^2} \text{ radians per orbit.}$$

For Earth, $J_2 = 0.00108$. The orbit characteristics $a$ and $i$ can be "tuned" ($e$ is essentially zero) so that the precession rate is 1/365th of a degree per Earth-day. This why tons of Earth satellites (e.g. Terra, Aqua) have altitude ~700 km and $i = \sim 98^\circ$.

(4) Mars' oblateness is ~ 0.00196 is greater than Earth's and it's radius ~3390 km. To put Mars Global Surveyor in a "2pm", sun-synchronous LMO, MGS's areocentric orbit is inclined at 92.9° to Mars' equator, at an altitude of ~380 km.

(5) The Hubble Space Telescope orbits at alt. = 560 km, $i = 28.5^\circ$; it's orbit precesses about Earth at a higher rate, and in the opposite direction than those of Terra and Aqua.

The last orbital element is a time index; after all, we also need to know where in it's orbit the satellite is. A common one is the "time of periapsis passage".

**Launching satellites and space probes**-
- Launch sites near the equator are preferred, because you can launch into most prograde inclinations of orbit from there.
- You also get a boost of ~05. km/s of "$\Delta v$" from the Earth's spin
- Retrograde orbits ($i > 90^\circ$) always require an out-of-plane maneuver which is very expensive, $\Delta v$-wise.
Interplanetary transfer-

Probes to other planets obviously have to get from Earth to their destinations. This process can be broken down into 4 steps: Earth escape, transfer (a.k.a. ‘cruise phase’), planetary rendezvous, and orbit insertion or landing. Each of these steps requires the expenditure of ‘Δv’.

To leave Earth:

$$E = 0 = \frac{1}{2} v_{esc}^2 - G \frac{M}{r_{earth}} \quad v_{esc} = \sqrt{\frac{2GM}{r_{earth}}}$$

= ~11.2 km/s. This step only gets us clear of Earth and into a heliocentric orbit essentially the same as Earth’s, with a heliocentric velocity:

$$v_o = \sqrt{\frac{GM_{sun}}{R_{earth-sun}}}$$

= 29.8 km/s. The next step is getting into a 'transfer' orbit. Obviously, the transfer orbit has to intersect both the spacecraft's original orbit and the orbit of the planetary target. It also has to be planned so that when the spacecraft arrives at it's target's orbit, the planet is, in fact, there.
The most efficient transfer orbit is called the Hohmann Transfer:

![Hohmann transfer](image)

Hohmann transfer (dashed) to superior planet (left) and inferior planet (right).

(6) A large number of probes have been sent to Mars; after Earth-escape most are put on basically Hohmann trajectories to Mars. To proceed we need this Preliminary Fact: Combining eqns (4), (6), (7), it can be shown that

\[
E = -\frac{GM}{2a}
\]  

(9)

which relates the energy of an orbit to its size.

Applying eqn (9) and (7) to the perihelion of an Earth-Mars Hohmann orbit:

\[
E = -G \frac{M_{\text{sun}}}{r_{\text{earth-sun}} + r_{\text{mars-sun}}} = \frac{1}{2} v_1^2 - G \frac{M_{\text{sun}}}{r_{\text{earth-sun}}}
\]

This can be solved for \(v_1\) which turns out to be about 32.7 km/s. Thus the spacecraft must be given a \(\Delta v\) of \(~3\) km/s.

Exactly the same methods can be used to find the speed of Mars in its orbit (24.1 km/s), and the speed of the spacecraft when it reaches Mars (21.5 km/s). This means that, when the spacecraft reaches Mars' orbit, Mars is actually 'catching up to' it. Thus to rendezvous with Mars it needs about 2.6 km/s more speed, for a total of \(~5.6\) km/s just for the transfer (not including the \(>11.2\) km/s required to escape Earth).

(7) To enter martian orbit about Mars probe must lose more speed still, otherwise it will simply describe a hyperbolic path around Mars and "re-
To actually land on Mars it has to lose the equivalent of Mars' escape velocity of \(~5 \text{ km/s}\). Therefore, it takes a total of at least \(21.8 \text{ km/s}\) of \(\Delta v\) to land a probe on Mars.

Not all of this has to be via rocket burning, however. Note that there ARE other planets in the solar system . . . Also, notice that orbit insertion or landing actually require de-celeration, relative to Mars . . .

Gravity assist-
   -Cassini, Messenger

Gravity assisted trajectories of Cassini (left) and Messenger (right).

Braking into orbit or landing-
   -Retro-rockets
   -Aerobraking: Magellan, Mars Global Surveyor
   -Parachutes: Viking, Mars Rovers, Huygen
   -Impact: Pathfinder, Mars Exploration Rovers