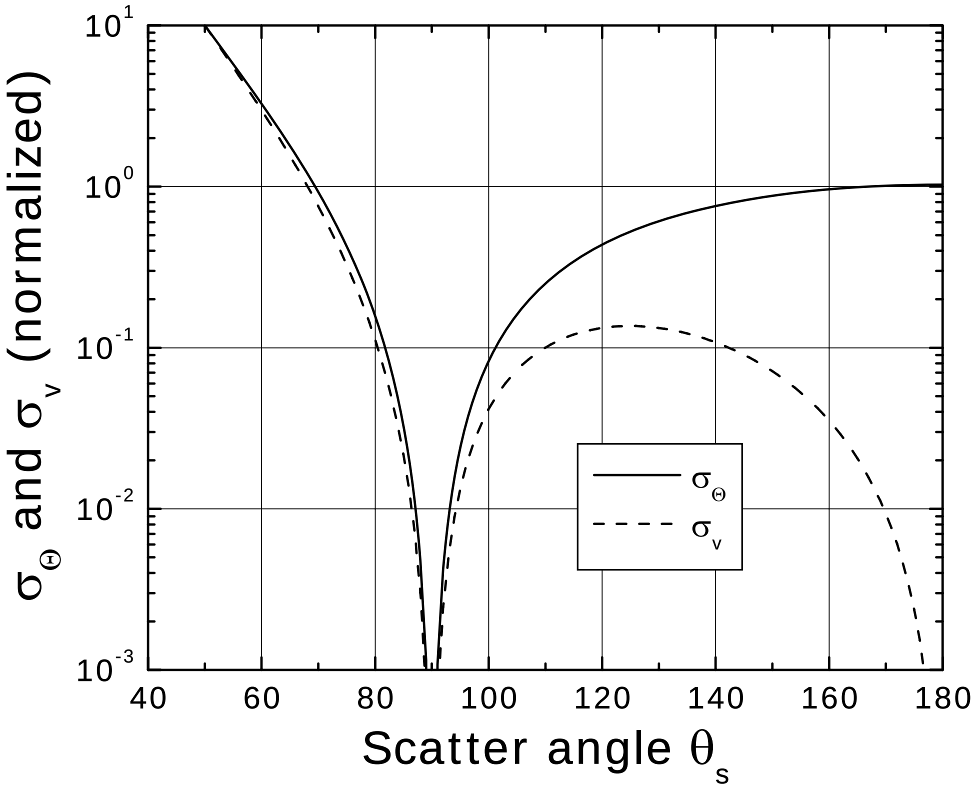

Sodar Data and Scintillometer Data Obtained from the UPOS Project "Optical Turbulence" and Applied to Study the Turbulence Structure in the Atmospheric Surface Layer

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Content

Abstract

In this report the use of Sodar and scintillometer data to analyze the structure of atmospheric turbulence, in particular, to determine structure functions of second-order and higher-order, and to investigate energy-inertial interactions, intermittency effects or extended self-similarity is described and discussed. It is shown that any straight-forward research on atmospheric turbulence is not possible on the basis of sole Sodar and scintillometer observations. Therefore, it is argued to consider additional measurements to relate such observations to typical quantities of the atmospheric surface layer (ASL) and the whole atmospheric boundary layer (ABL).

In chapters 2 to 4 of this report, dimensional scaling regarding the structure functions of second-order and higher-orders for the fields of the longitudinal and lateral velocity components and the temperature in the inertial range and the far dissipation range as well as the intermediate transition range under locally isotropic conditions at sufficiently large Reynolds numbers is described in a strictly mathematical manner by using the dimensional π-invariants analysis. It is shown that in the case of the asymptotic solutions for either the inertial range or the far dissipation range only one π number occurs that has to be considered as a non-dimensional universal constant. This π number may be determined theoretically or/and empirically. In the case of the transition range two π numbers occur. Consequently, a universal function is established that has to be derived theoretically or/and empirically, too. Here, Batchelor's classical interpolation formula for the turbulence structure of the velocity field and the empirical one of Stolovitzki et al. (1993), both may serve as a universal functions, are compared with the results from a numerical solution of Kolmogorov's structure equation of the velocity field for the transition range. It is shown that these interpolation formulae match the numerical results in an excellent manner. In the case of the temperature field results from the numerical solution of Yaglom's structure equation for the transition range are presented and discussed in a similar manner.

In chapters 5 to 7 of this report, various instances of non-dimensionalization are represented that are relevant to the turbulence transfer across the ASL, in particular, (a) Monin-Obukhov scaling for forced convective conditions, and (b) Prandtl-Obukhov-Priestley scaling for free convective condition. It is shown that in the case of only one π number the derived equations are really applicable when this π number can be determined empirically or/and theoretically. Such a π number can also be considered as a non-dimensional universal constant. This is true in the instances (a) of neutral stratification, when in the case of momentum transfer a logarithmic wind profile prevails and the π number is equal to the reciprocal of the von Kármán constant, and (b) of the Prandtl-Obukhov-Priestley scaling for free convective condition, for which flux-gradient relationships and the temperature variance relationship can be derived. Furthermore, it is shown that in the case of Monin-Obukhov scaling generally two π numbers occur. In such kind of scaling local similarity functions depending on the Obukhov number, ζ, can be established that may be considered as universal functions within the framework of the various similarity hypotheses. Unfortunately, these universal functions cannot be quantified by the dimensional π-invariants analysis so that their determination by empirical or/and theoretical work is indispensable. This is true for forced convective conditions for which local similarity functions of the flux-gradient relationships for the transfer of momentum, sensible heat and water vapor as well as local similarity functions of the energy dissipation and the normalized variances of wind components, potential temperature and specific humidity can be derived empirically, as done by several authors during the last five decades.

Since eddy fluxes of momentum, sensible heat and water vapor directly measured by fast response sensors like sonic anemometers and IR water vapor sensors are not always available, an aerodynamic profile method is described and applied to predict, at least, the friction velocity and the vertical components of the eddy fluxes of sensible and latent heat by using vertical profiles of the mean values of horizontal wind speed, potential temperature, and specific humidity obtained from slow-response measurements. The eddy flux results provided by different parameterization schemes, however, substantiate that great uncertainty exists in the prediction of the eddy fluxes of sensible and latent heat. With respect to climate prediction especially for high latitude regions like the Arctic, this uncertainty seems to be too large. Thus, more fast-response measurements are necessary to improve such parameterization schemes and to minimize their uncertainty.

1. Introduction

Sodar and scintillometer equipments can be applied to measure the structure parameter of the refractive index, ${C_n}^2$, for acoustic and electromagnetic waves propagating to the atmosphere. These ${C_n}^2$ data, however, cannot be used in a straight-forward manner to analyze the structure of atmospheric turbulence, in particular, to determine structure functions of second-order and higher-order, for investigating energy-inertial interactions, intermittency effects or extended self-similarity (see, e.g., Sreenivasan and Antonia, 1997; Katul, et al., 1997). This means that any straight-forward research on atmospheric turbulence is not possible on the basis of sole Sodar and scintillometer observations. Therefore, it is indispensable to consider additional measurements to relate such observations to typical quantities of the atmospheric surface layer (ASL) and the whole atmospheric boundary layer (ABL). Such quantities are, for instance, the Reynolds stress vector, $\boldsymbol{\tau} = \overline{\rho\, u''\, w''}\,\mathbf{i} + \overline{\rho\, v''\, w''}\,\mathbf{j}$, and the vertical components of the eddy flux densities (usually called the vertical eddy fluxes) of sensible heat, $H = c_{p,0}\,\overline{\rho\, w''\,\Theta''}$, and water vapor, $W = \overline{\rho\, w''\, q''}$, as well as the variances of wind speed, temperature and specific humidity. Here, u'', v'', and w'' are the fluctuations of the wind vector components in x-, y- and z-direction, with respect to their mean values $\hat{u}$, $\hat{v}$, and $\hat{w}$. Furthermore, Θ'' is the fluctuation of the potential temperature with respect to the mean potential temperature, $\hat{\Theta}$, and q'' is the fluctuation of the specific humidity, $q = \rho_w / \rho$, with respect to the mean specific humidity, $\hat{q}$, where ρ_w is the partial density of water vapor, and ρ the density of air. Note that Hesselberg's (1926) density-weighted average, $\hat{f} = \overline{\rho\, f} / \overline{\rho}$, is applied, where f represents u, v, w, Θ, as well as q, and the overbar ($\overline{}$) designates the Reynolds' mean customarily used. As illustrated, the turbulent fluxes correspond to covariance terms. All these covariance and variance data have to be applied to relate ${C_n}^2$ to the thermal structure of the ABL and the embedded ASL.

It is well-known that the structure parameter, ${C_\Theta}^2$, for temperature is closely identified with ${C_n}^2$ (e.g., Panofsky and Dutton, 1984; Kaimal and Finnigan, 1994) usually be expressed by an empirical relation like (e.g., Wesely, 1976; Kohsiek et al., 2002, Hartogensis et al., 2002)

$$ {C_n}^2 = \left(\frac{A_\Theta}{\hat{\Theta}} \right)^2 {C_\Theta}^2 \tag{1.1} $$

with $A_\Theta = -7.749 \cdot 10^{-5}\, \bar{p}/\hat{\Theta}$, where $\bar{p}$ is the mean pressure. (Note that the effect of water vapor is ignored in this equation because it is usually considered as negligible.) The physical background of such a relation can be explained as follows:

The theory of sound scattering from locally isotropic and homogeneous turbulence using the Born approximation provides the following expression for the acoustic differential scattering cross-section area per unit volume (per unit solid angle; Batchelor, 1957; Tatarskii, 1961, 1971; Neff and Coulter, 1986)

$$\sigma(\theta_s) = \frac{1}{8} k^4 \cos^2(\theta_s) \left\{ \frac{\phi_T(K)}{T_0^{\,2}} + \frac{\cos^2(\theta_s/2)\, E(K)}{\pi\, c_0^{\,2}\, K^2} \right\} \quad . \tag{1.2}$$

Here, θ_s is the angle at the scattering volume from the transmitter beam axis to the receiver, k is the acoustic wave number, $K = 2\,k \sin(\theta_s/2)$ the Bragg-scattering wave number, T_0 is the local temperature, c_0 is the corresponding speed of sound, $\phi_T(K)$ is the isotropic three-dimensional spectral density of temperature, and $E(K)$ is the turbulent kinetic energy. The principal features of this equation are: (1) only $\phi_T(K)$ contributes to backscatter, (2) neither $\phi_T(K)$ nor $E(K)$ contributes at $\theta_s = 90°$, and (3) the spectra are evaluated at $K = 2\,k \sin(\theta_s/2)$.

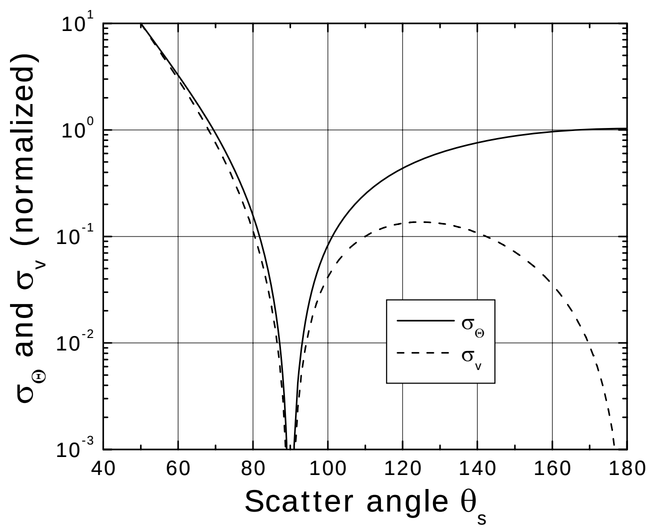

Figure 1: Contributions to the scattering cross sections of separate effects of temperature fluctuations, σ_Θ, and velocity fluctuations, σ_v (with reference to Little, 1972).

Unfortunately, the relation (1.2) between the scattering cross section and the spectra of temperature and energy at wave number K does not provide the direct information about the profiles of wind and temperature. From the knowledge in the structure functions and the energy cascade of inertial range of turbulence and the governing equations for turbulent moments as well as from experiments we can derive an equation for the scattering cross section that reads (e.g., Neff and Coulter, 1986)

$$\sigma(\theta_s) = \frac{1}{8} k^4 K^{-\frac{11}{3}} \cos^2(\theta_s) \left\{ \frac{0.033\, C_\Theta^{\,2}}{T_0^{\,2}} + \frac{\cos^2(\theta_s/2)}{\pi\, c_0^{\,2}} 0.76\, C_v^{\,2} \right\} \quad , \tag{1.3}$$

where $C_\Theta^{\,2}$ and $C_v^{\,2}$ are the structure parameter of temperature and velocity, respectively. The first term in the braces represents the contribution to the scattering at angle θ_s owing to temperature inhomogeneities, and the second term represents the contribution from turbulent velocity fluctuations. The contributions to the scattering cross section provided by each term in Eq. (1.3) are illustrated in Figure 1 in a normalized manner.

It is also well-known that $C_\Theta^{\,2}$ is strongly dependent on the geometric height z. (The same is true in the case of $C_v^{\,2}$.) For forced convective conditions, for instance, we have (e.g., Kaimal and Finnigan, 1994)

$$\frac{C_\Theta^{\,2}\, z^{\frac{2}{3}}}{\Theta_*^{\,2}} = \begin{cases} 5\left(1 + 6.4\,|\zeta|\right)^{-\frac{2}{3}} & \text{for } \zeta \le 0 \\ 5\left(1 + 3\,\zeta\right) & \text{for } \zeta \ge 0 \end{cases} \quad , \tag{1.4}$$

where $\zeta = z/L$ is the Obukhov number, L is the Obukhov stability length, Θ_* is the temperature scale defined by $\Theta_* = -H/\left(c_{p,0}\,\bar{\rho}\,u_*\right)$. Equation (1.4) is invalid under free convective conditions, i.e. for large negative values of ζ (very unstable stratification). Under such conditions $C_\Theta^{\,2}$ is given by (e.g., Panofsky and Dutton, 1984)

$$C_\Theta^{\,2} = 2.5 \left(\frac{H}{c_{p,0}\,\bar{\rho}} \right)^{\frac{4}{3}} \left(\frac{g}{\Theta_m} \right)^{-\frac{2}{3}} z^{-\frac{4}{3}} \quad . \tag{1.5}$$

Here, $c_{p,0}$ the specific heat at constant pressure, g is the acceleration of gravity, and Θ_m is a mean temperature characteristic for the layer under study. Since $C_\Theta^{\,2}$ can be related to the scatter angle, Eq. (1.5) customarily serves to determine the sensible heat flux from Sodar observations, where free convective conditions have to be assumed. Obviously, knowledge of the Obukhov number (or the Richardson number) is essential to differentiate between forced and free convective conditions for applying either Eq. (1.4) or Eq. (1.5). To allow an insight into structure functions for the flow velocity and temperature fields and the related structure parameters, these structure functions are generally discussed in chapters 2 to 4 by using dimensional scaling.

The Obukhov stability length is defined by

$$L = \frac{u_*^{\,2}}{\kappa \dfrac{g}{\Theta_m} \left(\Theta_* + 0.608\, \Theta_m\, q_*\right)} \quad . \tag{1.6}$$

Thus, the vertical eddy fluxes of sensible heat and water vapor, and the friction velocity defined by $u_* = +\sqrt{|\boldsymbol{\tau}|/\bar{\rho}}$ is required to determine that stability length. Here, $\kappa \cong 0.4$ is the von Kármán constant, and q_* is the humidity scale defined by $q_* = -W/(\bar{\rho}\, u_*)$.

It is obvious that eddy fluxes of momentum, sensible heat and water vapor directly measured by sonic anemometers and IR water vapor sensors must be available at the ARM (Atmospheric Radiation Measurement Program) site at Barrow/Alaska. Without such eddy flux data, the Obukhov stability length cannot be determined. Consequently, Eq. (1.4) would become rather worthless. Furthermore, an application of Eq. (1.5) without any Quality Assurance/Quality Control (QA/QC) activities regarding ζ would not provide any reliable eddy flux result. This would mean that a typical application of surface layer relationships fails completely. To avoid such a failure, a micrometeorological method is describe in this report that can be applied to predict u_*, H, and W as well as the variances of w, Θ, and q from the vertical profiles of the mean values of horizontal wind, potential temperature, and specific humidity measured by the ARM site at Barrow. This method is based on Kramm (1989) and Kramm et al. (1996), where new aspects regarding free convective conditions are included. This micrometeorological method is described in detail in chapters 5 to 7 and the numerical representation is given in Appendix D.

2. Structure functions for the velocity and temperature fields

To quantitatively describe the structure of the velocity field in the inertial range (sometimes called the inertial subrange, see, e.g., Batchelor, 1953) and the far dissipation range as well as the intermediate transition range under locally isotropic conditions at sufficiently large Reynolds numbers, Kolgomorov (1941a,b) introduced the so-called structure function of second-order

$$D_{ij}(r) = D_{ij}(M, M') = \overline{\left(v_i(M') - v_i(M)\right)\left(v_j(M') - v_j(M)\right)} \quad \text{for } i = 1, 2, 3 \text{ and } j = 1, 2, 3 \quad , \qquad (2.1)$$

where $v_i(M)$ and $v_j(M)$ as well as $v_i(M')$ and $v_j(M')$ represent the i^{th} and j^{th} components of the velocity vector measured at points M and M', and the overbar characterizes averaging. The distance between the two points M and M' is denoted by r. As argued by him, this distance should be much smaller than the integral scale Λ, i.e. $r << \Lambda$. Since under these conditions the motion of the eddies (with sizes much smaller than Λ) is assumed to be homogeneous and isotropic, this structure function is considered as an invariant tensor function with respect to the vector $\overrightarrow{MM'}$ (e.g., Obukhov and Yaglom, 1951; Rotta, 1972). The elements of this velocity-correlation tensor of second-rank may be expressed by

$$D_{ij}(r) = A(r)\,\xi_i\,\xi_j + B(r)\,\delta_{ij} \quad \text{for } i = 1, 2, 3 \text{ and } j = 1, 2, 3 \quad , \qquad (2.2)$$

where ξ_i, $i = 1, 2, 3,$ are the components of the vector $\overrightarrow{MM'}$, and δ_{ij} is Kronecker's delta. Hence, the distance between M and M' is given by $r = \left(\xi_1{}^2 + \xi_2{}^2 + \xi_3{}^2\right)^{1/2}$. Setting $v_i = v_j = v_d$, where v_d is the longitudinal velocity component (i.e., parallel to the vector $\overrightarrow{MM'}$), and $v_i = v_j = v_n$, where v_n is the lateral velocity component (i.e., perpendicular to the vector $\overrightarrow{MM'}$), yields then (Obukhov and Yaglom, 1951, Rotta, 1972)

$$D_{ij}(r) = \frac{D_{2d}(r) - D_{2n}(r)}{r^2}\, \xi_i\, \xi_j + D_{2n}(r)\, \delta_{ij} \quad \text{for } i = 1, 2, 3 \text{ and } j = 1, 2, 3 \quad . \tag{2.3}$$

Here,

$$D_{2\alpha}(r) = \overline{\left(v_\alpha(M') - v_\alpha(M)\right)^2} = 2\left(B(0) - B_{2\alpha}(r)\right) \tag{2.4}$$

is the structure function of second-order for the velocity components parallel ($v_\alpha = v_d$) and perpendicular ($v_\alpha = v_n$) to the vector $\overrightarrow{MM'}$, respectively. The expression $B_{2\alpha}(r) = \overline{v_\alpha(M')\, v_\alpha(M)}$ is the correlation function, where $B(0) = B_{2\alpha}(0)$.

Based on his first similarity hypothesis that the probability distribution functions of locally isotropic turbulence only depends on the mean dissipation rate, $\bar{\varepsilon}$, considered as independent of r, and the kinematic viscosity, ν, Kolmogorov (1941a,b) showed that this structure function can further be written as

$$D_{2\alpha}(\xi) = \left(\bar{\varepsilon}\, \nu\right)^{\frac{1}{2}}\, \varphi_{2\alpha}(\xi) \quad , \tag{2.5}$$

where $\varphi_{2\alpha}(\xi)$ is a universal function and $\xi = r/\eta$ with $\eta = \left(\nu^3 / \bar{\varepsilon}\right)^{1/4}$, usually called Kolmogorov's local scale of turbulence. He derived following asymptotic solutions for the far dissipation range ($r << \eta$)

$$\varphi_{2\alpha}(\xi) = \delta_{2\alpha}\, \xi^2 \tag{2.6}$$

with

$$\delta_{2\alpha} = \begin{cases} \dfrac{1}{15} & \text{for } \alpha = d \\[2ex] \dfrac{2}{15} & \text{for } \alpha = n \end{cases} \quad , \tag{2.7}$$

and for the inertial range ($\eta << r << \Lambda$)

$$\varphi_{2\alpha}(\xi) = \chi_{2\alpha}\, \xi^{\frac{2}{3}} \tag{2.8}$$

with

$$\chi_{2\alpha} = \begin{cases} \chi_{2d} & \text{for } \alpha = d \\ \\ \frac{4}{3}\chi_{2d} & \text{for } \alpha = n \end{cases} \quad , \tag{2.9}$$

where χ_{2d} is a non-dimensional universal constant that amounts to $\chi_{2d} = \frac{27}{55}\Gamma\left(\frac{1}{3}\right)\beta = 1.315\,\beta$. Here, $\beta = 1.44 \pm 0.06$ (see Appendix B) is customarily called the Kolmogorov constant (e.g., Rotta, 1972; Sreenivasan and Antonia, 1997), and $\Gamma(\ldots)$ is the gamma function. This $2/3$ power law was deduced by Kolmogorov (1941a) and Obukhov (1941) independently from each other. The former obtained it on the basis of his second similarity hypothesis that in the inertial range the probability distribution functions solely depends on $\bar{\varepsilon}$, i.e., they are independent of ν. The latter derived it by computing the balance of energy distribution over the spectrum in the inertial range.

On the basis of the von Kármán-Howarth equation for locally isotropic turbulence of an incompressible fluid (e.g., von Kármán and Howarth, 1938; Landau and Lifshitz, 1959; Barenblatt, 1979; Stanišić, 1985) Kolmogorov (1941b) argued that the transition between the inertial range and the far dissipation range can be described by the structure equation

$$6\,\nu\,\frac{dD_{2d}(r)}{dr} - D_{3d}(r) = \frac{4}{5}\,\bar{\varepsilon}\,r \quad , \tag{2.10}$$

where

$$D_{3\alpha}(r) = \overline{\left(v_\alpha(M') - v_\alpha(M)\right)^3} \tag{2.11}$$

is the third-order structure function. He showed that in the case of the asymptotic solution for the far dissipation range the first term on the left hand side of Eq. (2.10) amounts to

$$6\,\nu\,\frac{dD_{2d}(r)}{dr} \cong \frac{4}{5}\,\bar{\varepsilon}\,r \quad , \tag{2.12}$$

i.e., the third moment, $D_{3d}(r)$, is negligible because of the r^3 behavior of $D_{3d}(r)$ when r approaches to zero (Landau and Lifshitz, 1959). On the other hand, in the case of the inertial range, the first term on the left hand side of Eq. (2.10) becomes

$$6\,\nu\,\frac{dD_{2d}(r)}{dr} = 4\,\nu\,\chi_{2d}\,\bar{\varepsilon}^{\,2/3}\,r^{-1/3} \quad . \tag{2.13}$$

Compared to $\left|4\,\bar{\varepsilon}\,r/5\right|$, this term is rather negligible, and, hence, we have

$$D_{3d}(r) \cong -\frac{4}{5}\,\bar{\varepsilon}\,r \quad . \tag{2.14}$$

Obviously, Eq. (2.10) has to be solved numerically. A first attempt was made by Obukhov and Yaglom (1951) using Kolgomorov's (1941b) assumption that in both the far dissipation range and the inertial range the so-called skewness (e.g., Kolmogorov, 1941b; Stanišić, 1985; Katul et al., 1997),

$$S(r) = \frac{D_{3\alpha}(r)}{\{D_{2\alpha}(r)\}^{\frac{3}{2}}} \quad , \tag{2.15}$$

is independent of r. Equation (2.15) customarily serves to replace the third-order structure function in Kolmogorov's structure equation (2.10).

Analogous to Kolmogorov's (1941b) structure equation of the velocity field for the transition between the inertial range and the far dissipation range, Yaglom (1949) derived the following equation (see also Katul et al., 1997; Chassaing et al., 2002)

$$2\,\alpha_T\,\frac{dD_{2T}(r)}{dr} - D_{d,2T}(r) = \frac{4}{3}\,\bar{N}\,r \quad . \tag{2.16}$$

Here, α_T is the thermal diffusivity, and the second and third moments are given by

$$D_{2T}(r) = D_{2T}(M, M') = \overline{(T(M') - T(M))^2} \tag{2.17}$$

and

$$D_{d,2T}(r) = D_{d,2T}(M, M') = \overline{(v_d(M') - v_d(M))(T(M') - T(M))^2} \quad , \tag{2.18}$$

respectively. This mixed structure function can be expressed by $D_{d,2T}(r) = S_T\,\{D_{2d}(r)\}^{1/2}\,D_{2T}(r)$. The quantity, S_T, may be considered as the statistical analogue to the skewness S and is also called the mixed skewness (e.g., Sreenivasan and Antonia, 1997). It is defined by (e.g., Yaglom, 1949; Katul et al., 1997; Sreenivasan and Antonia, 1997)

$$S_T = \frac{D_{d,2T}(r)}{\{D_{2d}(r)\}^{\frac{1}{2}}\,D_{2T}(r)} \quad . \tag{2.19}$$

As in the case of Eq. (2.10), Yaglom's (1949) structure equation has to be solved numerically, too. Such a solution was qualitatively sketched by Obukhov (1949).

2.1 Complete and incomplete similarity in the inertial range

As pointed out, for instance, by Frisch (1995), Yaglom (2001) as well as Lumley and Yaglom (2001), Kolmogorov´s theory was based on very clear and convincing physical arguments expressed in the form of two hypotheses concerning the mechanisms generating small-scale turbulent fluctuations. When Kolmogorov developed his similarity theory no experimental data were available for the purpose of comparison, i.e., all results and conclusions based on this similarity theory had the character of pure predictions. First attempts to compare the predictions of Kolmogorov's theory with the results of specially posed experiments apparently confirmed this theory. However, as stated, for instance, by Lumley and Yaglom (2001), measurements of small scale turbulent velocity fluctuations performed in the laboratory by Batchelor and Townsend (1949) and ASL measurements carried out by researchers working at the Moscow Institute of Atmospheric Physics in the late 1950 revealed data clearly contradicting Kolmogorov's prediction. These empirical results underlined that Kolmogorov's (1941a) similarity hypotheses were only approximately true for the velocity field. In addition, soon after the papers of Kolmogorov (1941a,b) and Obukhov (1941) were published, L. D. Landau (as cited by Kolmogorov, 1962) noticed that the similarity hypothesis for the inertial range did not take into account a circumstance which directly arises from the assumption of the essential accidental and random character of the mechanism of transfer of energy from the coarser vortices to the finer; with the increase of the ratio Λ/η, the variation of the dissipation of energy should increase without limit. Based on Obukhov's (1962) arguments on the spatial variation of this energy dissipation, Kolmogorov (1962) formulated a generalized version of his second similarity hypothesis of 1941 that results in

$$D_{2\alpha}(r) = \chi_{2\alpha} * r^{\frac{2}{3}} \bar{\varepsilon}^{\frac{2}{3}} \left(\frac{r}{\Lambda}\right)^{k} = \frac{\chi_{2\alpha} *}{\left(\bar{\varepsilon}\,\Lambda\right)^{k}} \left(r\,\bar{\varepsilon}\right)^{\vartheta_2} \tag{2.20}$$

and

$$S(r) = S_0 \left(\frac{\Lambda}{r}\right)^{\frac{3}{2}k} \quad . \tag{2.21}$$

Here, $\chi_{2\alpha} *$ and S_0 depends on the macrostructure of the flow, $\vartheta_2 = (2 + 3\,k)/3$, and $k \cong 0.04$ is a universal constant related to log $\overline{(\log \varepsilon_r)^2} = A - 9\,k \log(\Lambda/r)$, where A also depends on the macrostructure of the flow. Obviously, the exponent ϑ_2 only differs slightly from the classical value $2/3$. Kolmogorov also derived an expression for the structure function of arbitrary order,

$$|D_{m\alpha}(r)| = \left|\overline{(v_\alpha(M') - v_\alpha(M))^m}\right| = \chi_{m\alpha} * \left(r\,\bar{\varepsilon}\right)^{\frac{m}{3}} \left(\frac{r}{\Lambda}\right)^{\frac{1}{2}k\,m\,(3-m)} = \frac{\chi_{m\alpha} *}{\left(\bar{\varepsilon}\,\Lambda\right)^{\frac{1}{2}k\,m\,(3-m)}} \left(r\,\bar{\varepsilon}\right)^{\vartheta_m} \tag{2.22}$$

with $\vartheta_m = m\left(2 + 3\,k\,(3 - m)\right)/6$. For $m = 2$, it provides Eq. (2.20), and for $m = 3$ we obtain Eq. (2.14), i.e., this equation holds, even though the refinement of Komogorov´s (1941b) original similarity hypothesis is considered (e.g., Kolmogorov, 1962; Stolovitzky and Sreenivasan, 1993; Sreenivasan and Antonia, 1997; Cerutti and Meneveau, 1998; Chassaing et al., 2002; Kurien, 2003).

Recognizing Kolmogorov's (1962) refinement, Barenblatt (1979, 1996) postulated that under locally isotropic conditions the structure function of second-order for the velocity field in the inertial range can be hypothesized by $F\left(D_{2\alpha}(r), \bar{\varepsilon}, r, \eta, \Lambda\right) = 0$ (see Eq. (A1)). Obviously, the number of the dimensional quantities is $k = 5$ and the table of fundamental dimensional quantities is given by

	$D_{2\alpha}(r)$	$\bar{\varepsilon}$	r	η	Λ
Length	2	2	1	1	1
Time	- 2	- 3	0	0	0

that leads to the dimensional matrix

$$G = \begin{Bmatrix} 2 & 2 & 1 & 1 & 1 \\ -2 & -3 & 0 & 0 & 0 \end{Bmatrix} \quad . \tag{2.23}$$

According to Eq. (A9), the system of linear equations can be written as

$$\begin{Bmatrix} 2 & 2 & 1 & 1 & 1 \\ -2 & -3 & 0 & 0 & 0 \end{Bmatrix} \begin{Bmatrix} \alpha_{1,i} \\ \alpha_{2,i} \\ \alpha_{3,i} \\ \alpha_{4,i} \\ \alpha_{5,i} \end{Bmatrix} = \{\mathbf{0}\} \quad \text{for } i = 1, 2, 3 \quad . \tag{2.24}$$

The rank of the dimensional matrix is $R = 2$. Thus, we have $p = k - R = 3$ independent π numbers that can be derived from

$$\left.\begin{array}{llllllll} 2\,\alpha_{1,i} & +2\,\alpha_{2,i} & +\alpha_{3,i} & +\alpha_{4,i} & +\alpha_{5,i} & = & 0 \\ -2\,\alpha_{2,i} & -3\,\alpha_{2,i} & & & & = & 0 \end{array}\right\} \quad \text{for } i = 1, 2, 3 \quad . \tag{2.25}$$

Choosing $\alpha_{1,1} = 1$, $\alpha_{4,1} = 0$, $\alpha_{5,1} = 0$, $\alpha_{1,2} = 0$, $\alpha_{1,3} = 0$, $\alpha_{3,2} = 1$, $\alpha_{5,2} = 0$, $\alpha_{3,3} = 1$, and $\alpha_{4,3} = 0$ yields $\alpha_{2,1} = -2/3$, $\alpha_{3,1} = -2/3$, $\alpha_{2,2} = 0$, $\alpha_{2,3} = 0$, $\alpha_{4,2} = -1$ and $\alpha_{5,3} = -1$. Our choice results in (see Eq. (A2))

$$\pi_1 = \left(D_{m\alpha}(r)\right)^1 \bar{\varepsilon}^{-\frac{2}{3}} r^{-\frac{2}{3}} \eta^0 \Lambda^0 = D_{m\alpha}(r)\left(r\,\bar{\varepsilon}\right)^{-\frac{2}{3}} \quad , \tag{2.26}$$

$$\pi_2 = \left(D_{m\alpha}(r)\right)^0 \, \bar{\varepsilon}^{\,0} \, r^1 \, \eta^{-1} \, \Lambda^0 = \frac{r}{\eta} \tag{2.27}$$

and

$$\pi_3 = \left(D_{m\alpha}(r)\right)^0 \, \bar{\varepsilon}^{\,0} \, r^1 \, \eta^0 \, L^{-1} = \frac{r}{\Lambda} \quad . \tag{2.28}$$

According to Eq. (A3b), we may write

$$D_{2\alpha}(r) = \pi_1 \, r^{\frac{2}{3}} \, \bar{\varepsilon}^{\frac{2}{3}} \, \varphi_{2\alpha}(\pi_2, \pi_3) = \pi_1 \, r^{\frac{2}{3}} \, \bar{\varepsilon}^{\frac{2}{3}} \, \varphi_{2\alpha}\left(\frac{r}{\eta}, \frac{r}{\Lambda}\right) \quad . \tag{2.29}$$

Obviously, for the inertial range, $\eta << r << \Lambda$, and sufficiently large Reynolds numbers, we have $r/\eta \to \infty$, but $r/\Lambda \to 0$. As argued by Barenblatt (1979; 1996), in the classical Kolmogorov-Obukhov theory an assumption is implicitly made that is equivalent to the assumption that there is a finite non-zero limit of $\varphi_{2\alpha}(r/\eta, r/\Lambda)$ as $r/\eta \to \infty$ and $r/\Lambda \to 0$, i.e., there is a complete self-similarity in both non-dimensional parameters r/η and r/Λ. Therefore, for $\eta << r << \Lambda$ one obtains the two-thirds law (see also Eq. (2.8)) that was deduced by Kolmogorov (1941a) and Obukhov (1941) independently from each other, where $\chi_{2\alpha} = \pi_1 \, \varphi_{2\alpha}(\infty, 0)$.

The existence of complete similarity in the parameter r/Λ for small values of r/Λ, however, seems to be doubtful owing to the so-called intermittency effect. Intermittency occurs when the spatial distribution of the energy transfer rate towards smaller vortices is non-uniform. as already postulated by L.D. Landau and recognized by Kolmogorov (1962) and Obukhov (1962). Thus, it is quite reasonable to assume complete similarity in the parameter r/η for $r/\eta >> 1$ and incomplete similarity in the parameter r/Λ for $r/\Lambda << 1$ so that we have, as $r/\eta \to \infty$ and $r/\Lambda \to 0$ (Barenblatt, 1979; 1996),

$$\varphi_{2\alpha}\left(\frac{r}{\eta}, \frac{r}{\Lambda}\right) \cong C \left(\frac{r}{\Lambda}\right)^k \quad . \tag{2.30}$$

Introducing this equation into Eq. (2.29) yields then Eq. (2.20), where $\chi_{2\alpha}{}^* = \pi_1 \, C$.

Under certain additional assumptions, Barenblatt and Goldenfeld (1995) found for the inertial range

$$D_{2\alpha}(r) = \left(A_0 + \frac{A_1}{\log Re}\right) \left(r \, \bar{\varepsilon}\right)^{\frac{2}{3}} \left(\frac{r}{\Lambda}\right)^{\frac{k}{\log Re}} \quad , \tag{2.31}$$

where A_0, A_1, and k are universal constants, and Re is the Reynolds number. Obviously, compared to the exponent $2/3$ to which r enters Eq. (2.31), the correction by $k/\log Re$ is not substantial for very large Reynolds numbers. As argued by Barenblatt (1996), this small effect can hardly be observed at present. Moreover, Eq. (2.31) states that the correction of the Kolmogorov constant appears to be inversely proportional to $\log \mathrm{Re}$, which is consistent with the experimental data of Praskovsky and Oncley (1994).

Based on these findings, it is assumed that complete similarity as implicitly involved in dimensional scaling can be used as a prerequisite to describe the turbulence structure of the flow and temperature fields in the inertial range and the far dissipation range under locally isotropic conditions at sufficiently large Reynolds numbers, where principles of the dimensional π-invariants analysis (see Appendix A) is applied to describe the procedure of non-dimensionalization, on which this kind of scaling is based, in a strictly mathematical manner. It is shown that in the case of the asymptotic solution for either the inertial range or the far dissipation range only one π number occurs that has to be considered as a non-dimensional universal constant. This π number may be determined theoretically or/and empirically. In the case of the transition from the inertial range to the far dissipation range two π numbers occur. Consequently, a universal function is established that has to be derived theoretically or/and empirically, too. Here, Batchelor's classical interpolation formula for the turbulence structure of the flow field and the empirical one of Stolovitzki et al. (1993), both may serve as a universal functions, are compared with the results from a numerical solution of Kolmogorov's structure equation of the velocity field for the whole transition range. It is shown that these interpolation formulae match the numerical results in an excellent manner. In the case of the temperature field results from the numerical solution of Yaglom's structure equation for the whole transition range are presented and discussed, too.

3. Structure functions for the velocity field

Kolmogorov's (1941a) first similarity hypothesis states that under locally isotropic conditions the turbulence structure of the velocity field for the inertial range and the far dissipation range as well as the intermediate transition range at sufficiently large Reynolds numbers can be described by $F\left(\mathrm{D}_{m\alpha}(r), \bar{\varepsilon}, r, \nu\right) = 0$. (see Eq. (A1)). Here,

$$\mathrm{D}_{m\alpha}(\mathrm{M}, \mathrm{M}') = \overline{\left(v_\alpha(\mathrm{M}') - v_\alpha(\mathrm{M})\right)^m} \tag{3.1}$$

is, again, the structure function of arbitrary order $m = 2, 3, \ldots$ for the velocity components parallel ($v_\alpha = v_d$) and perpendicular ($v_\alpha = v_n$) to the vector $\overrightarrow{\mathrm{MM}'}$, respectively. These velocity components have to be measured at the points M and M', where the distance between these two points is denoted by r, i.e., $\mathrm{D}_{m\alpha}(r) = \mathrm{D}_{m\alpha}(\mathrm{M}, \mathrm{M}')$. As mentioned before, this distance should be much smaller than the integral scale, Λ, i.e. $r \ll \Lambda$. Beyond this scale isotropic turbulence must not be expected. Obviously, the number of the dimensional quantities is $k = 4$ and the table of fundamental dimensional quantities is given by

	$\mathrm{D}_{m\alpha}(r)$	$\bar{\varepsilon}$	r	ν
Length	m	2	1	2
Time	- m	- 3	0	- 1

The dimensional matrix, therefore, reads

$$\mathbf{G} = \begin{Bmatrix} m & 2 & 1 & 2 \\ -m & -3 & 0 & -1 \end{Bmatrix} \quad . \tag{3.2}$$

According to Eq. (A9), the system of linear equations can be written as

$$\begin{Bmatrix} m & 2 & 1 & 2 \\ -m & -3 & 0 & -1 \end{Bmatrix} \begin{Bmatrix} \alpha_{1,i} \\ \alpha_{1,i} \\ \alpha_{1,i} \\ \alpha_{1,i} \end{Bmatrix} = \{\mathbf{0}\} \quad \text{for } i = 1, 2 \quad . \tag{3.3}$$

The rank of the dimensional matrix is $R = 2$. Thus, we have $p = k - R = 2$ independent π numbers that can be derived from

$$\left.\begin{array}{lllllll} m\,\alpha_{1,i} & +2\,\alpha_{2,i} & +\alpha_{3,i} & +2\,\alpha_{4,i} & = & 0 \\ -m\,\alpha_{1,i} & -3\,\alpha_{2,i} & & -\alpha_{4,i} & = & 0 \end{array}\right\} \quad \text{for } i = 1, 2 \quad . \tag{3.4}$$

Choosing $\alpha_{1,1} = 1$, $\alpha_{3,1} = 0$, $\alpha_{1,2} = 0$, and $\alpha_{3,2} = 1$ yields $\alpha_{2,1} = -m/4$, $\alpha_{4,1} = -m/4$, $\alpha_{2,2} = 1/4$, and $\alpha_{4,2} = -3/4$. Our choice results in (see Eq. (A2))

$$\pi_1 = \left(D_{m\alpha}(r)\right)^1 \bar{\varepsilon}^{-\frac{m}{4}} r^0 \nu^{-\frac{m}{4}} = D_{m\alpha}(r)\left(\bar{\varepsilon}\,\nu\right)^{-\frac{m}{4}} \tag{3.5}$$

and

$$\pi_2 = \left(D_{m\alpha}(r)\right)^0 \bar{\varepsilon}^{\frac{1}{4}} r^1 \nu^{-\frac{3}{4}} = r\left(\frac{\bar{\varepsilon}}{\nu^3}\right)^{\frac{1}{4}} = \frac{r}{\eta} = \xi \quad , \tag{3.6}$$

where $\eta = \left(\nu^3 / \bar{\varepsilon}\right)^{1/4}$ is, again, Kolmogorov's local scale of turbulence. According to Eq. (A3b), we may write

$$\pi_1 = D_{m\alpha}(r)\left(\bar{\varepsilon}\,\nu\right)^{-\frac{m}{4}} = \varphi_{m\alpha}(\pi_2) = \varphi_{m\alpha}(\xi) \tag{3.7}$$

or

$$D_{m\alpha}(\xi) = \left(\bar{\varepsilon}\,\nu\right)^{\frac{m}{4}} \varphi_{m\alpha}(\xi) \quad , \tag{3.8}$$

where $\xi = r/\eta$. Empirical evidence for structure functions of higher orders can be found, for instance, in Stolovitzky et al. (1993). For $m = 2$, we obtain Eq. (2.5). The third-order structure function is given by (e.g., Stanišić, 1985)

$$D_{3\alpha}(\xi) = \left(\bar{\varepsilon}\,\nu\right)^{\frac{3}{4}}\,\varphi_{3\alpha}(\xi) \quad . \tag{3.9}$$

As expressed by Eq. (2.15), these structure functions of second-order and third-orders can be used to determine the skewness. Introducing Eqs. (2.5) and (3.9) into Eq. (2.15) yields then

$$S = \frac{\varphi_{3\alpha}(\xi)}{\left\{\varphi_{2\alpha}(\xi)\right\}^{\frac{3}{2}}} \quad , \tag{3.10}$$

i.e., the skewness only depends on the universal functions of the velocity field of second-order and third-order.

There were several attempts to determine such universal functions of various orders for describing the transition between the inertial range and the far dissipation range (e.g., Obukhov and Yaglom, 1951; Batchelor, 1951; Benzi et al, 1993; Stolovitzky and Sreenivasan, 1993; Stolovitzky et al., 1993; Sirovich et al., 1994; Meneveau, 1996; Gaudin et al., 1998). It is obvious that such structure functions (or the universal functions) have to fulfill the matching conditions of the asymptotic solutions for both the inertial range and the far dissipation range.

3.1 Asymptotic solutions for the inertial range

Kolgomorov's (1941a) second similarity hypothesis regarding the turbulence structure of the inertial range reads $F\left(D_{m\alpha}(r), \bar{\varepsilon}, r\right) = 0$ (see Eq. (A1)), i.e., viscosity plays no role. This can be assumed when the condition $\eta << r << \Lambda$ is fulfilled. Obviously, the number of the dimensional quantities is $k = 3$. The table of fundamental dimensions given by

	$D_{m\alpha}(r)$	$\bar{\varepsilon}$	r
Length	m	2	1
Time	- m	- 3	0

that leads the dimensional matrix

$$\mathbf{G} = \begin{Bmatrix} m & 2 & 1 \\ -m & -3 & 0 \end{Bmatrix} \quad . \tag{3.11}$$

According to Eq. (A9), the system of linear equations reads

$$\begin{Bmatrix} m & 2 & 1 \\ -m & -3 & 0 \end{Bmatrix} \begin{Bmatrix} \alpha_{1,1} \\ \alpha_{2,1} \\ \alpha_{3,1} \end{Bmatrix} = \{\mathbf{0}\} \quad . \tag{3.12}$$

Again, the rank of the dimensional matrix is $R = 2$. Hence, we have $p = k - R = 1$ independent π number that can be derived from

$$\left.\begin{array}{rrrcl} m\,\alpha_{1,1} & +2\,\alpha_{2,1} & +\alpha_{3,1} & = & 0 \\ -m\,\alpha_{1,1} & -3\,\alpha_{2,1} & & = & 0 \end{array}\right\} \quad . \tag{3.13}$$

Choosing $\alpha_{1,1} = 1$ yields $\alpha_{2,1} = -m/3$ and $\alpha_{3,1} = -m/3$. Our choice leads to (see Eq. (A2))

$$\pi_1 = \left(D_{m\alpha}(r)\right)^1 \varepsilon^{-\frac{m}{3}} r^{-\frac{m}{3}} \tag{3.14}$$

or

$$D_{m\alpha}(r) = \pi_1\, \varepsilon^{\frac{m}{3}} r^{\frac{m}{3}} \quad . \tag{3.15}$$

For $m = 2$, we obtain the so-called two-thirds law (see also Eq. (2.8))

$$D_{2\alpha}(r) = \pi_1\, \varepsilon^{\frac{2}{3}} r^{\frac{2}{3}} \quad . \tag{3.16}$$

The non-dimensional universal constant, π_1, is given by Eq. (2.9). Equation (3.16) may also be written as

$$D_{2\alpha}(r) = C_{2\alpha}{}^2\, r^{\frac{2}{3}} \quad . \tag{3.17}$$

Here, $C_{2\alpha}{}^2 = \chi_{2\alpha}\, \varepsilon^{2/3}$ is called the structure parameter for the respective velocity component v_α, i.e., this structure parameter is defined for the inertial range only. If we consider the universal function for the energy dissipation in the ASL (see, e.g., Eq. (6.62)),

$$\frac{\kappa\,(z-d)}{u_*{}^3}\,\varepsilon = \Phi_\varepsilon(\zeta) \quad , \tag{3.18}$$

the structure parameter can be related to $\Phi_\varepsilon(\zeta)$ by

$$C_{2\alpha}{}^2 = \chi_{2\alpha}\, u_*{}^2 \left\{ \frac{\Phi_\varepsilon(\zeta)}{\kappa\,(z-d)} \right\}^{\frac{2}{3}} \quad , \tag{3.19}$$

where $z - d$ serves as a characteristic length scale. For diabatic stratification, Wyngaard and Coté (1971) as well as Kaimal et al. (1972) recommended

$$\Phi_\varepsilon(\zeta) = \begin{cases} \left(1 + 0.5\,|\,\zeta\,|^{2/3}\right)^{3/2} & \text{for} -2 \le \zeta \le 0 \\ \\ \left(1 + 2.5\,|\,\zeta\,|^{3/5}\right)^{3/2} & \text{for } 0 \le \zeta \le 2 \end{cases} \quad . \tag{3.20}$$

Obviously, $\Phi_\varepsilon(0)$ is equal to unity for neutral stratification.

According to Eq. (3.14), the asymptotic solution for the third-order moment reads (e.g., Stanišić, 1985)

$$D_{3\alpha}(r) = \pi_1\, \bar{\varepsilon}\, r = \chi_{3\alpha}\, \bar{\varepsilon}\, r \quad , \tag{3.21}$$

i.e., the third-order structure function linearly depends on r. By considering Eq. (2.14), the non-dimensional universal constant can be identified with $\pi_1 = \chi_{3\alpha} = -4/5$. Thus, in the case of the inertial range the skewness (defined by Eq. (2.15)),

$$S = \frac{\chi_{3\alpha}}{\chi_{2\alpha}{}^{\frac{3}{2}}} \quad , \tag{3.22}$$

is independent of the distance r.

By considering Eq. (3.8), the asymptotic solution of the universal function, $\varphi_{m\alpha}(\xi)$, may be expressed by

$$\varphi_{m\alpha}(\xi) = \pi_1\, \xi^{\frac{m}{3}} \quad . \tag{3.23}$$

The universal functions of second-order is given by Eq. (2.8), and that of the third-order reads

$$\varphi_{3\alpha}(\xi) = \chi_{3\alpha}\, \xi \quad . \tag{3.24}$$

Note that the similarity hypothesis $F(E(k), \varepsilon, k) = 0$ that is equivalent with Kolmogorov's (1941a) second similarity hypothesis regarding the turbulence structure of the inertial range leads to the well-known $-5/3$ power law,

$$E(k) = \beta\, \bar{\varepsilon}^{\frac{2}{3}}\, k^{-\frac{5}{3}} \quad , \tag{3.25}$$

where $E(k)$ is the spectral energy distribution and k is the wave number. Equation (3.25) was derived independently from each other by Kolmogorov (1941a), Obukhov (1941), Onsager (1945) and von Weizsäcker (1948).

3.2 Asymptotic solutions for the far dissipation range

To describe the asymptotic behavior of the universal function for distances $r << \eta$, we assume that the turbulence structure of the far dissipation range can hypothetically be expressed by $F\left(D_{m\alpha}(r), \bar{\varepsilon}/\nu, r\right) = 0$ (see Eq. (A1)). With this hypothesis it is recognized that viscosity becomes dominant in the far dissipation range. As before, the number of the dimensional quantities is $k = 3$. Now, the dimensional matrix reads

$$\mathbf{G} = \begin{Bmatrix} m & 0 & 1 \\ -m & -2 & 0 \end{Bmatrix} \quad . \tag{3.26}$$

It can be derived from the table of fundamental dimensions given by

	$D_{m\alpha}(r)$	$\bar{\varepsilon}/\nu$	r
Length	m	0	1
Time	- m	- 2	0

The system of linear equations reads then (see Eq. (A9))

$$\begin{Bmatrix} m & 0 & 1 \\ -m & -2 & 0 \end{Bmatrix} \begin{Bmatrix} \alpha_{1,1} \\ \alpha_{2,1} \\ \alpha_{3,1} \end{Bmatrix} = \{\mathbf{0}\} \quad . \tag{3.27}$$

As before, the rank of the dimensional matrix is $R = 2$. Hence, we have $p = k - R = 1$ independent π number that can be derived from

$$\left. \begin{array}{lllll} m\,\alpha_{1,1} & & +\alpha_{3,1} & = & 0 \\ -m\,\alpha_{1,1} & -\;2\,\alpha_{2,1} & & = & 0 \end{array} \right\} \quad . \tag{3.28}$$

Choosing $\alpha_{1,1} = 1$ yields $\alpha_{2,1} = -m/2$ and $\alpha_{3,1} = -m$. Thus, we obtain (see Eq. (A2))

$$\pi_1 = \left(D_{m\alpha}(k)\right)^1 \left(\frac{\bar{\varepsilon}}{\nu}\right)^{-\frac{m}{2}} r^{-m} \quad . \tag{3.29}$$

Rearranging this equation leads to

$$D_{m\alpha}(r) = \pi_1 \left(\frac{\bar{\varepsilon}}{\nu} \right)^{\frac{m}{2}} r^m \quad . \tag{3.30}$$

For $m = 2$, we obtain the well-known r^2 law for the far dissipation range (see also Eq. (2.6)),

$$D_{2\alpha}(r) = \pi_1 \frac{\bar{\varepsilon}}{\nu} r^2 \quad . \tag{3.31}$$

The non-dimensional universal constant, π_1, is given by Eq. (2.7).

Note that the similarity hypothesis $F\left(D_{2\alpha}(r), \bar{\varepsilon}/\nu^2, r\right) = 0$ would lead to

$$D_{2\alpha}(r) = \pi_1 \left(\frac{\bar{\varepsilon}}{\nu^2} \right)^2 r^6 \quad . \tag{3.32}$$

Here, the role of viscosity in the far dissipation range is addressed by $\bar{\varepsilon}/\nu^2$. Obviously, this expression is in complete contradiction to the expected r^2 law (see Eq. (3.31)). However, if we consider the similarity hypothesis $F\left(E(k), \bar{\varepsilon}/\nu^2, k\right) = 0$, we will obtain

$$E(k) = \pi_1 \left(\frac{\bar{\varepsilon}}{\nu^2} \right)^2 k^{-7} \tag{3.33}$$

that is well known as Heisenberg's (1948) -7 power law for the far dissipation range. The π number is identified with $\pi_1 = \kappa_H{}^2/4$, where $\kappa_H = 0.51 \pm 0.03$ is the Heisenberg constant. There is no evidence that supports the r^6 behavior in the far dissipation range. Consequently, addressing viscosity effects by $\bar{\varepsilon}/\nu^2$ might be the reason why Heisenberg's -7 power law fails when the wave number tends to infinity (see, e.g., Townsend, 1951; Hinze, 1959; Lumley and Panofsky, 1964; Stanišić, 1985; McComb, 1990).

Considering Eq. (3.30) leads to the asymptotic solution for the third-order structure function,

$$D_{3\alpha}(r) = \pi_1 \left(\frac{\bar{\varepsilon}}{\nu} \right)^{\frac{3}{2}} r^3 \tag{3.34}$$

with the non-dimensional constant $\pi_1 = \delta_{3\alpha}$. Thus, combining Eqs. (3.31) and (3.34) provides for the skewness (defined by Eq. (2.15)),

$$S = \frac{\delta_{3\alpha}}{\delta_{2\alpha}^{\frac{3}{2}}} \quad , \tag{3.35}$$

i.e., also in the case of the far dissipation range the skewness is independent of the distance r.

According to Eq. (3.8), the asymptotic solution of the universal function, $\varphi_{m\alpha}(\xi)$, may be expressed by

$$\varphi_{m\alpha}(\xi) = \pi_1 \, \xi^{m} \quad . \tag{3.36}$$

The universal functions of second-order is given by Eq. (2.6), and that of the third-order reads

$$\varphi_{3\alpha}(\xi) = \delta_{3\alpha} \, \xi^{3} \quad . \tag{3.37}$$

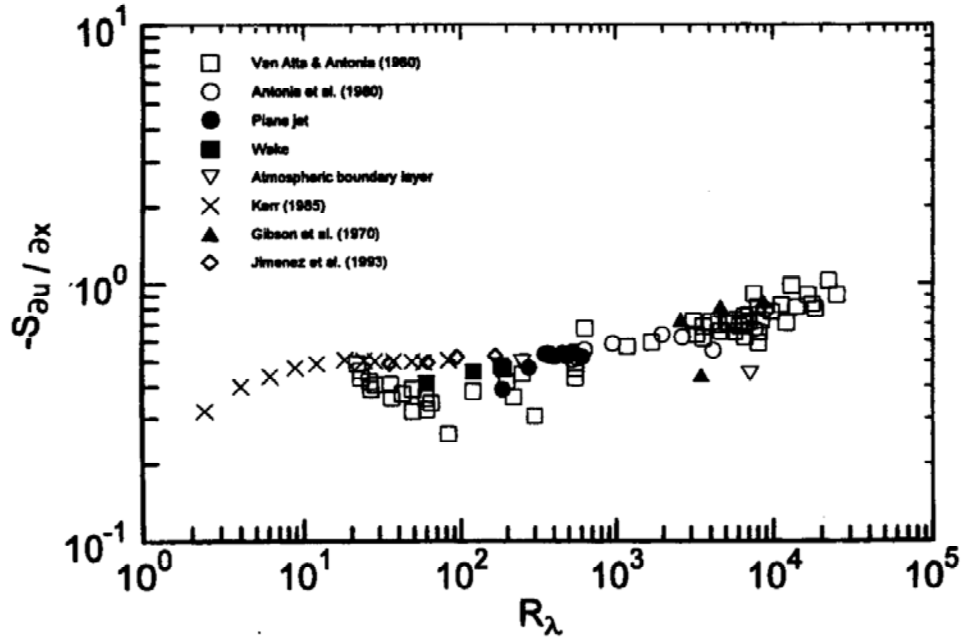

Figure 2: The Reynolds number (R_λ) variation of the skewness ($-S_{\partial u/\partial x}$) of the streamwise (or longitudinal) velocity derivative $\partial u/\partial x$ from different sources (from Sreenivasan and Antonia, 1997).

3.3 Interpolation formulae for the transition range

As mentioned before, Kolmogorov's (1941b) structure equation (2.10) can be used to quantitatively describe the transition from the inertial range to the far dissipation range (in the following we only consider the longitudinal velocity component v_d). According to Eq. (2.15), the third moment that occurs in this structure equation may be expressed by $D_{3d}(r) = S\{D_{2d}(r)\}^{3/2}$. Thus, we have

$$6\,\nu\,\frac{dD_{2d}(r)}{dr} - S\{D_{2d}(r)\}^{\frac{3}{2}} = \frac{4}{5}\,\bar{\varepsilon}\,r \quad . \tag{3.38}$$

Using Eq. (3.10) and Kolmogorov's local scale of turbulence (see also Eq. (2.5)) yields then

$$\frac{d\varphi_{2d}(\xi)}{d\xi} - \frac{S}{6}\{\varphi_{2d}(\xi)\}^{\frac{3}{2}} = \frac{2}{15}\,\xi \quad . \tag{3.39}$$

It was shown that the skewness, S, is independent of the distance r in both cases of the asymptotic solutions (see Eqs. (3.22) and (3.35)) as already argued by Kolmogorov (1941b). This finding may serve to justify the assumption of Obukhov and Yaglom (1951) that the skewness is independent of r for the whole transition range (theorem of constancy of skewness; Kolmogorov, 1962). In these premises, one can find for the inertial range: $\chi_{2d} = \{-4/(5\,S)\}^{2/3}$. Rearranging this expression yields then (see also Rotta, 1972)

$$S = -\frac{4}{5}\,\chi_{2d}^{-\frac{3}{2}} \cong -0.31 \quad . \tag{3.40}$$

Note that Kerr (1985, 1990) obtained from direct numerical simulation for isotropic turbulence $S = -0.4$ which agrees with that suggested by Landau and Lifshitz (1959). Stolovitzky et al. (1993) computed a mean value of $S = -0.38$ from fitted curves of second and third moments (see also Figure 8) that is in good agreement with $S = -0.39$ found by these authors in wind tunnel experiments. Katul et al. (1997) derived from field experiments: $S = -0.25$. As illustrated in Figure 2, there is a large scatter, where the results also suggest a dependency of the skewness on the Reynolds number. This result is reflected by Eq. (2.15), when Eq. (2.31) is used to express the second-order structure function. Nevertheless, combining Eq. (3.35) and (3.40) for $\alpha = d$ yields then

$$\delta_{3d} = -\frac{4}{5}\left(\frac{\delta_{2d}}{\chi_{2d}}\right)^{\frac{3}{2}} \quad . \tag{3.41}$$

If we introduce Eq. (3.40) into Eq. (3.39), we will obtain

$$\frac{d\varphi_{2d}(\xi)}{d\xi} + \frac{2}{15}\,\chi_{2d}^{-\frac{3}{2}}\{\varphi_{2d}(\xi)\}^{\frac{3}{2}} = \frac{2}{15}\,\xi \quad . \tag{3.42}$$

This equation may be solved numerically, where the initial condition has to be chosen by $\varphi_{2,d}(0) = 0$. Figure 3 illustrates the results of such a numerical solution. Also illustrated is the interpolation formula,

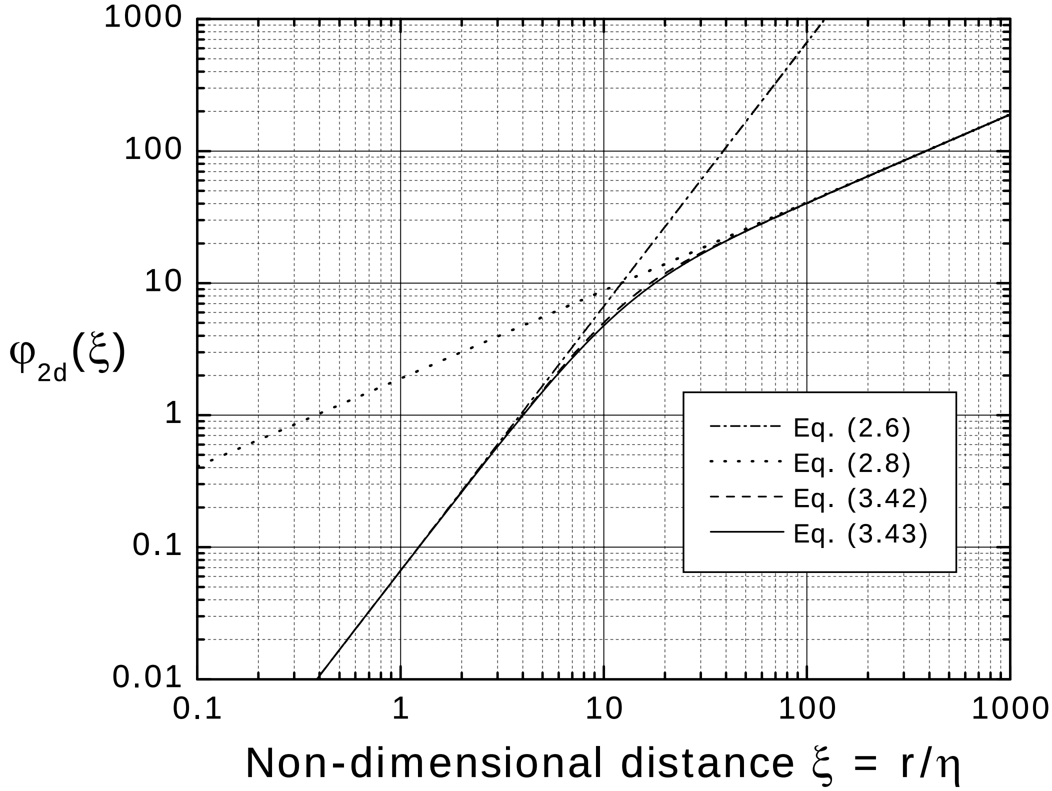

Figure 3: Computed universal functions $\varphi_{2d}(\xi)$ of the longitudinal velocity component and their asymptotic solutions plotted against the non-dimensional distance $\xi = r/\eta$.

$$\varphi_{2d}(\xi) = \frac{\delta_{2d}\,\xi^2}{\left(1+\left(\frac{\delta_{2d}}{\chi_{2d}}\right)^{\frac{3}{2}}\xi^2\right)^{\frac{2}{3}}} \quad , \tag{3.43}$$

proposed by Batchelor (1951) and recently considered by various authors (e.g., Stolovitzky et al., 1993; Sirovich et al., 1994; Meneveau, 1996), where Stolovitzky et al., 1993 generalized it to structure functions of arbitrary orders by fitting measured data. Because of

$$\varphi_{2d}(\xi) = \begin{cases} \chi_{2d}\,\xi^{\frac{2}{3}} & \text{for } \xi >> 1 \\ \delta_{2d}\,\xi^2 & \text{for } \xi << 1 \end{cases} \quad , \tag{3.44}$$

Batchelor's interpolation formula fulfils the requirements of the asymptotic solutions for both

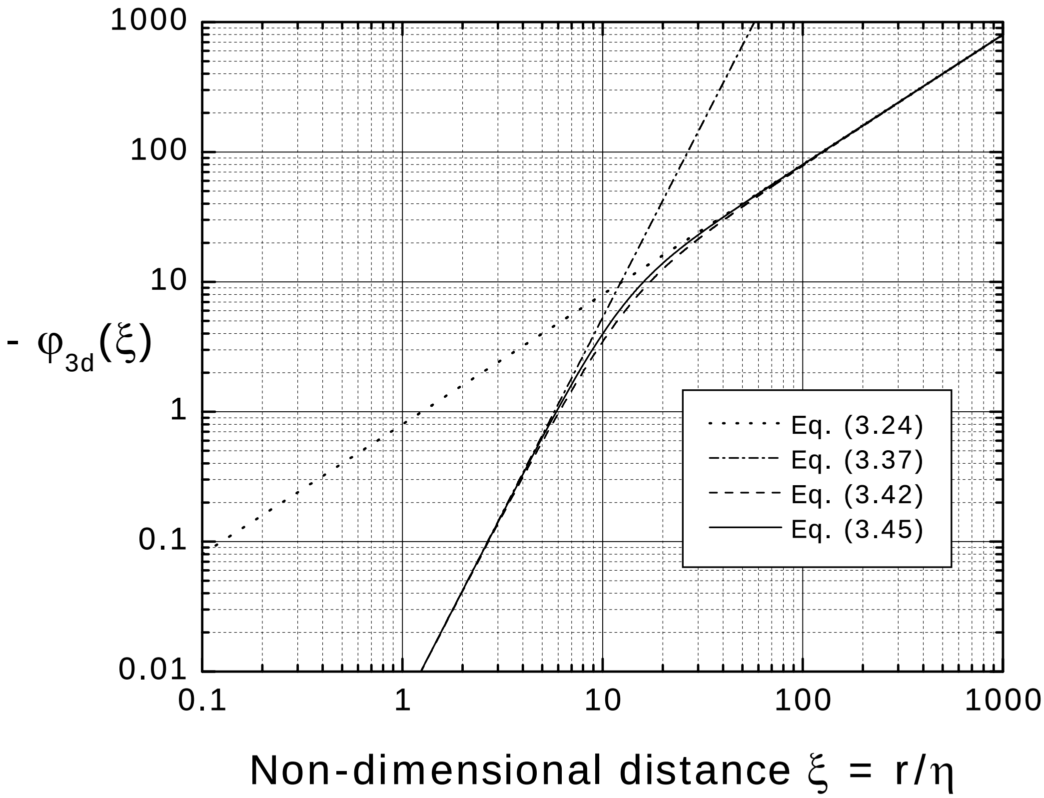

Figure 4: As in Figure 3, but for the universal functions $\varphi_{3d}(\xi)$.

$r << \eta$ and $\eta << r << \Lambda$ perfectly. This may be considered as a necessary condition for such an interpolation formula. It also matches the results obtained from the numerical solution of Eq. (3.42) in an excellent manner. A substantial agreement with these predictions may be considered as a sufficient condition for such an interpolation formula. Obviously, Batchelor's interpolation formula fulfils both the necessary condition and the sufficient condition, and it may serve as a second-order universal function.

Equation (3.10) may serve to compute $\varphi_{3d}(\xi)$. On the other hand, replacing ξ in Eq. (3.43) by $\xi^{m/2}$ as well as δ_{2d} and χ_{2d} by δ_{md} and χ_{md}, respectively, leads to

$$\varphi_{md}(\xi) = \frac{\delta_{md}\, \xi^{m}}{\left(1 + \left(\frac{\delta_{md}}{\chi_{md}}\right)^{\frac{3}{2}} \xi^{m}\right)^{\frac{2}{3}}} \tag{3.45}$$

with the well-known asymptotic solutions for the m-th order structure function gathered by

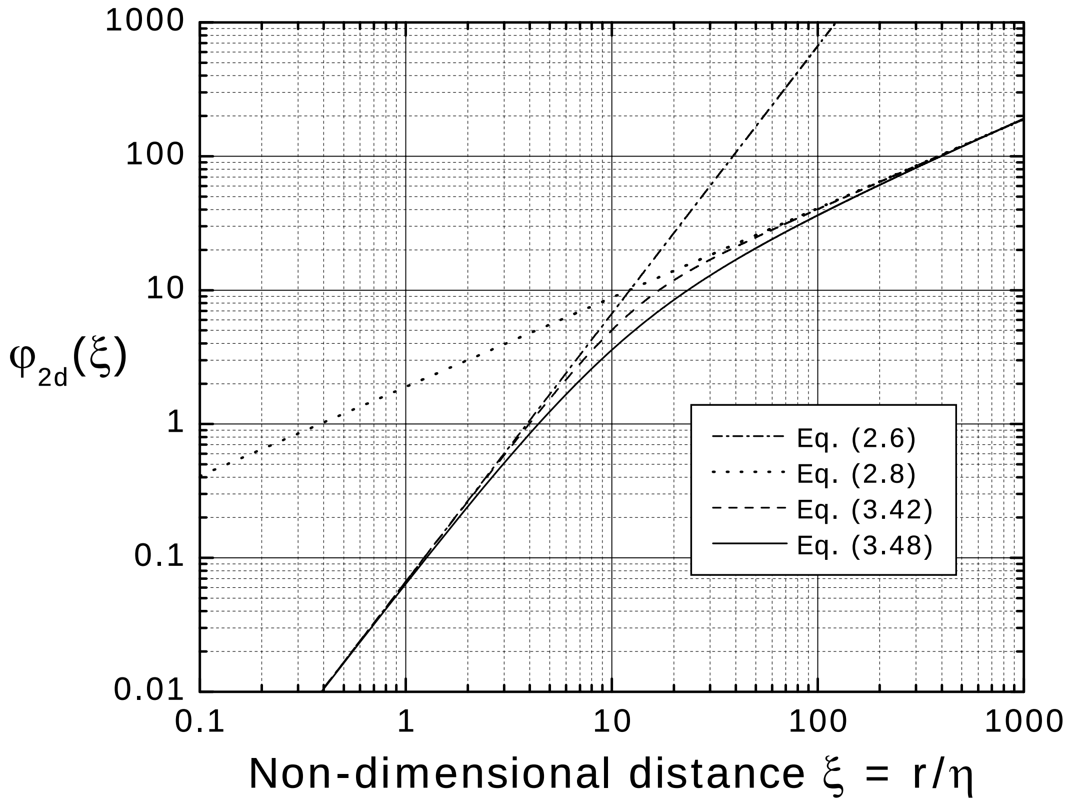

Figure 5: As in Figure 3, but for Eq. (3.48).

$$\varphi_{md}(\xi) = \begin{cases} \chi_{md}\, \xi^{\frac{m}{3}} & \text{for } \xi >> 1 \\ \delta_{md}\, \xi^{m} & \text{for } \xi << 1 \end{cases} \quad . \tag{3.46}$$

Such an extension of Batchelor's interpolation formula to higher-order moments seems to be reasonable and advantageous (see also Stolovitzky et al., 1993). Results obtained from the numerical solution of Eq. (3.42) in conjunction with Eq. (3.10) and those provided by Eq. (3.45) for $m = 3$ are illustrated in Figure 4. As expected, the results broadly agree. In the transition range, however, the interpolation formula slightly exceeds the numerical solution of Eq. (3.42).

The following instance shows that the necessary conditions may be fulfilled, but not the sufficient conditions: If we define $\mu = \xi^{2/3}$, the asymptotic solutions (2.6) and (2.8) can be gathered by

$$\varphi_{2d}(\mu) = \begin{cases} \delta_{2d}\, \mu^{3} & \text{for } r << \eta \\ \chi_{2d}\, \mu & \text{for } \eta << r << \Lambda \end{cases} \quad . \tag{3.47}$$

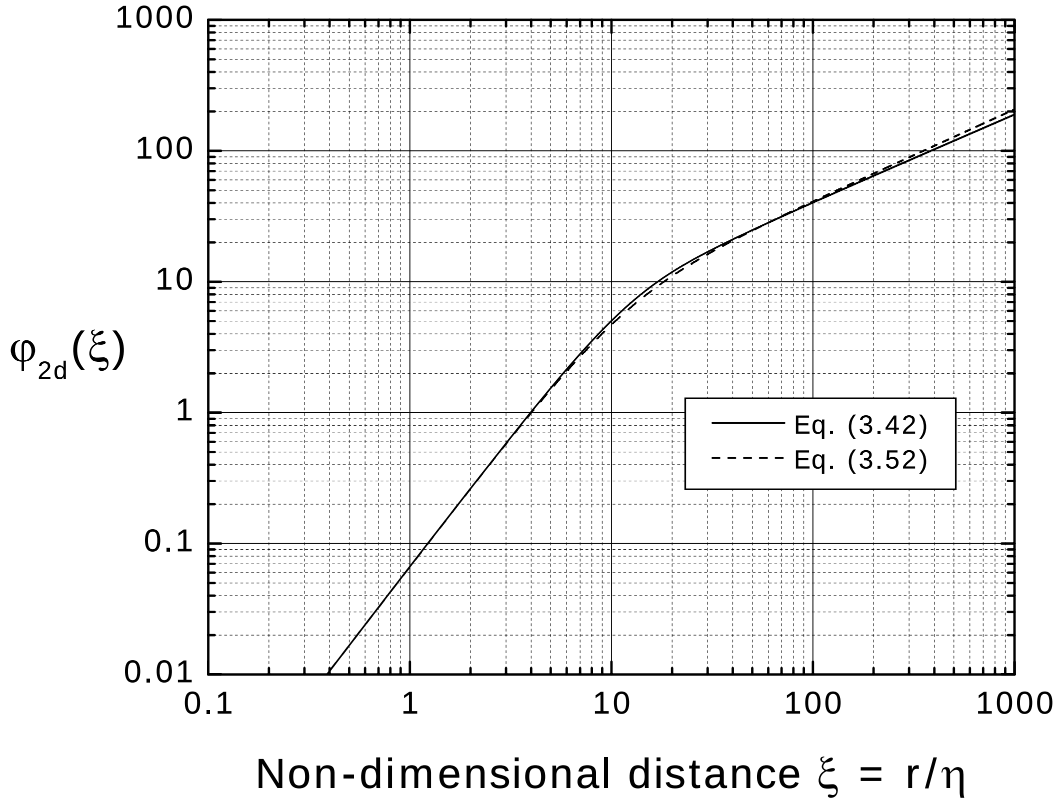

Figure 6: As in Figure 3, but for the interpolation formula of Stolovitzky et al. (1993).

An interpolation formula that fulfils these requirements for both $r \ll \eta$ and $\eta \ll r \ll \Lambda$, is given by

$$\varphi_{2d}(\mu) = \chi_{2d} * \left(\mu - \mu_0 \tanh \frac{\mu}{\mu_0} \right) \quad , \tag{3.48}$$

where μ_0 has to be determined. The μ^3-behavior of Eq. (3.48) for the range $r \ll \eta$ can be explained as follows: If we approximate the function $\tanh(\mu/\mu_0)$ by a Taylor series, we will obtain

$$\tanh \frac{\mu}{\mu_0} = \frac{\mu}{\mu_0} - \frac{1}{3}\left(\frac{\mu}{\mu_0}\right)^3 + \frac{2}{15}\left(\frac{\mu}{\mu_0}\right)^5 - \frac{17}{315}\left(\frac{\mu}{\mu_0}\right)^7 + \ldots \tag{3.49}$$

that converges for all values $|\mu/\mu_0| < \pi/2$ (where $\pi = 3.14\ldots$). For $r \ll \eta$, it is sufficient to consider the first two elements of this Taylor series. Thus, we have

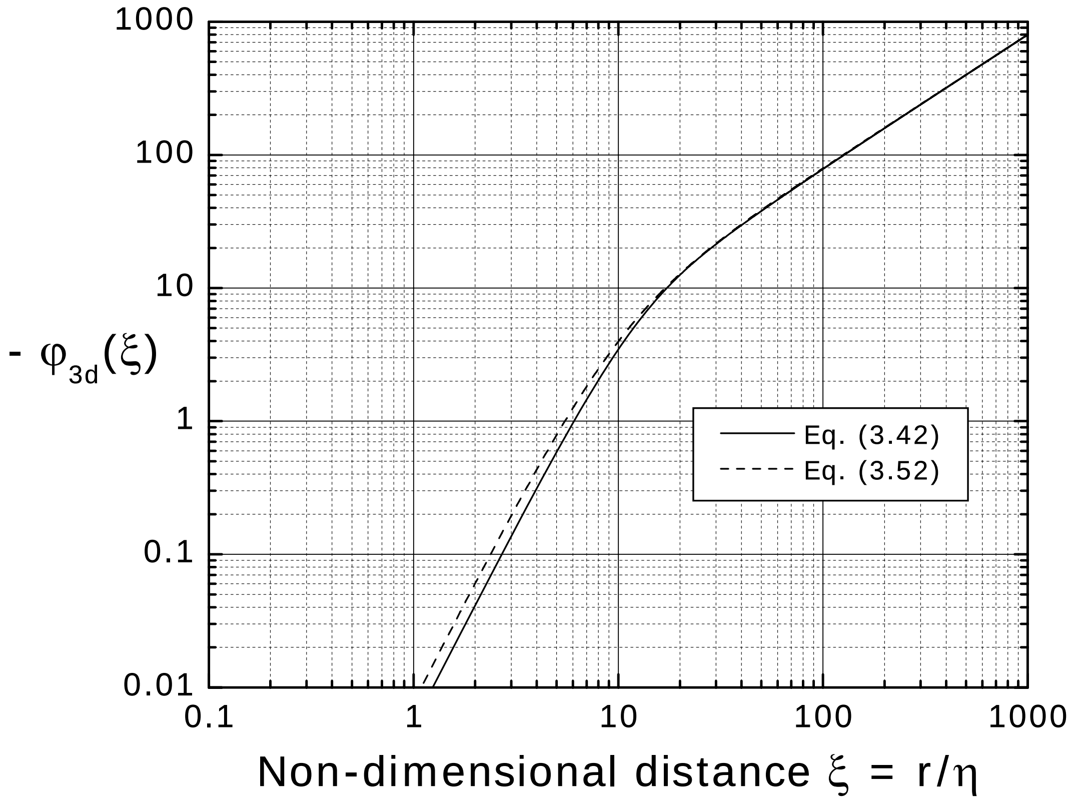

Figure 7: As in Figure 4, but for the interpolation formula of Stolovitzky et al. (1993).

$$\varphi_{2d}(\mu) = \chi_{2d}{}^* \left(\mu - \mu_0 \frac{\mu}{\mu_0} + \frac{\mu_0}{3} \left(\frac{\mu}{\mu_0} \right)^3 \right) = \frac{\chi_{2,d}{}^* \mu_0}{3} \left(\frac{\mu}{\mu_0} \right)^3 \cong \delta_{2,d}\, \mu^3 \quad . \tag{3.50}$$

The parameter μ_0 is then given by

$$\mu_0 = \left(\frac{\chi_{2d}{}^*}{3\,\delta_{2d}} \right)^{\frac{1}{2}} \quad . \tag{3.51}$$

For increasing values of μ the influence of the hyperbolic tangent decreases rapidly because $\tanh(x)$ tends to unity when x grows. In such a case we would have $\varphi_{2d}(\mu) = \chi_{2d}{}^*(\mu - \mu_0)$ with $\chi_{2d}{}^* = \lambda\,\chi_{2d}$, where λ serves to compensate the slight effect of μ_0 in this expression. On the other hand, considering the asymptotic solutions for the m-th order structure function gathered by Eq. (3.46), an equation similar to Eq. (3.48) could also be used to compute $\varphi_{md}(\xi)$. As illustrated in Figure 4, the interpolation formula touches both asymptotic solutions given by

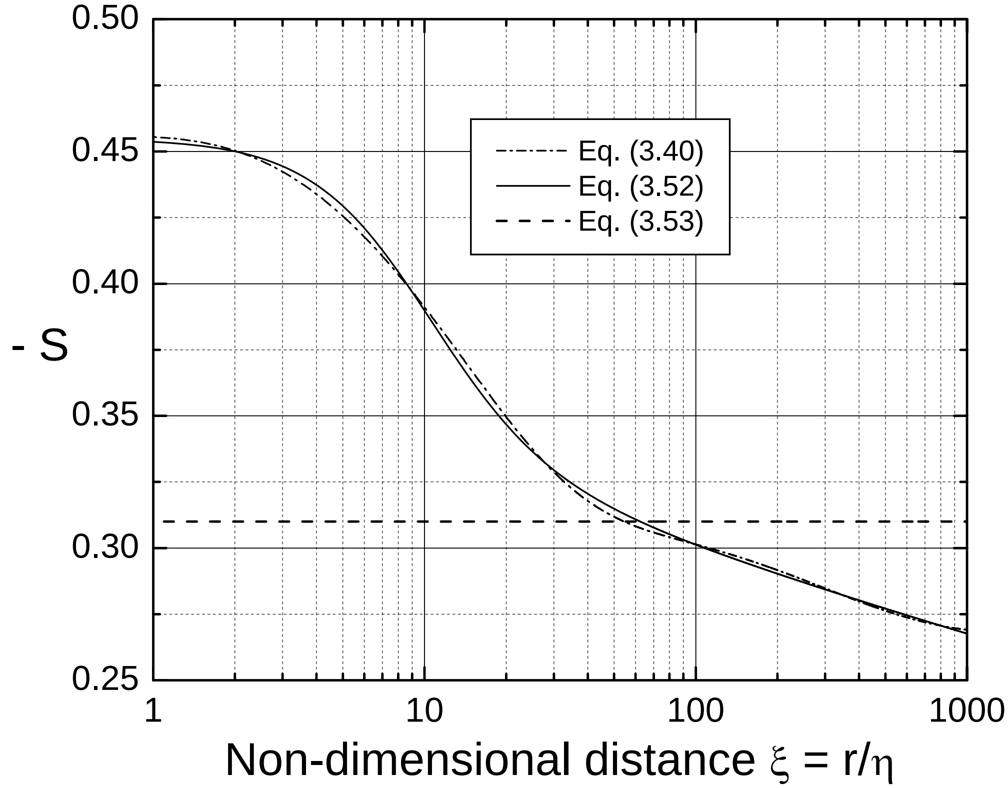

Figure 8: Skewness plotted against the non-dimensional distance $\xi = r/\eta$.

Eq. (3.47) in an excellent manner, i.e., it fulfils the necessary conditions. The sufficient condition, however, is hardly fulfilled because, as shown in Figure 5, the results provided by Eq. (3.48) do not match those obtained from the numerical solution of Eq. (3.42) with a sufficient degree of accuracy.

Figures 6 and 7 show results for the universal functions of second-order and third-order predicted by Eq. (3.42) and those delivered by the empirical interpolation formula of Stolovitzky et al. (1993) rearranged to

$$\varphi_{md}(\xi) = \left(\frac{\nu}{\varepsilon}\right)^{\frac{m}{2}} \frac{A_m\, \xi^{\,m}}{\left(1 + B_m\, \xi^{\,2}\right)^{C_m}} \quad . \tag{3.52}$$

The empirical quantities A_m, B_m, and C_m can be taken from their Table II. For the purpose of comparison, their quantity $(\nu/\varepsilon)^{m/2}$ was chosen in such a sense that it fits (a) for $m = 2$ the asymptotic solution in the far dissipation range and (b) for $m = 3$ the asymptotic solution in the inertial range. As illustrated in Figure 6, the results of the second-order universal functions are in a good agreement for $\xi < 100$. For larger values of ξ the interpolation formula (3.52) provides results that are somewhat higher than those of the numerical prediction. In the case of the third-order universal function, however, there is a excellent agreement for $\xi \geq 20$ (see Figure 7). For $\xi < 20$, the interpolation formula (3.52) provides notably higher values of $-\varphi_{3d}(\xi)$ than the numerical prediction. The reason for these differences can be attributed to the skewness. According to Eq. (3.40), it was chosen as constant. The interpolation formula (3.52), however, provides a skewness which clearly depends on ξ (see Figure 8). If this skewness is simply fitted by the exponential function

$$\left|S\right| = A_0 + A_1 \exp\left(-\frac{\xi}{\xi_1}\right) + A_2 \exp\left(-\frac{\xi}{\xi_2}\right) \quad , \tag{3.53}$$

with $A_0 = 0.2672$, $A_1 = 0.1548$, $A_2 = 0.0469$, $\xi_1 = 14.55$, and $\xi_2 = 306.1$, the solutions provided by Eq. (3.42) will well agree with the interpolation formula (3.52)within the margins of 2 percent when only the universal functions of second-order and third-order are considered (see Figure 9).

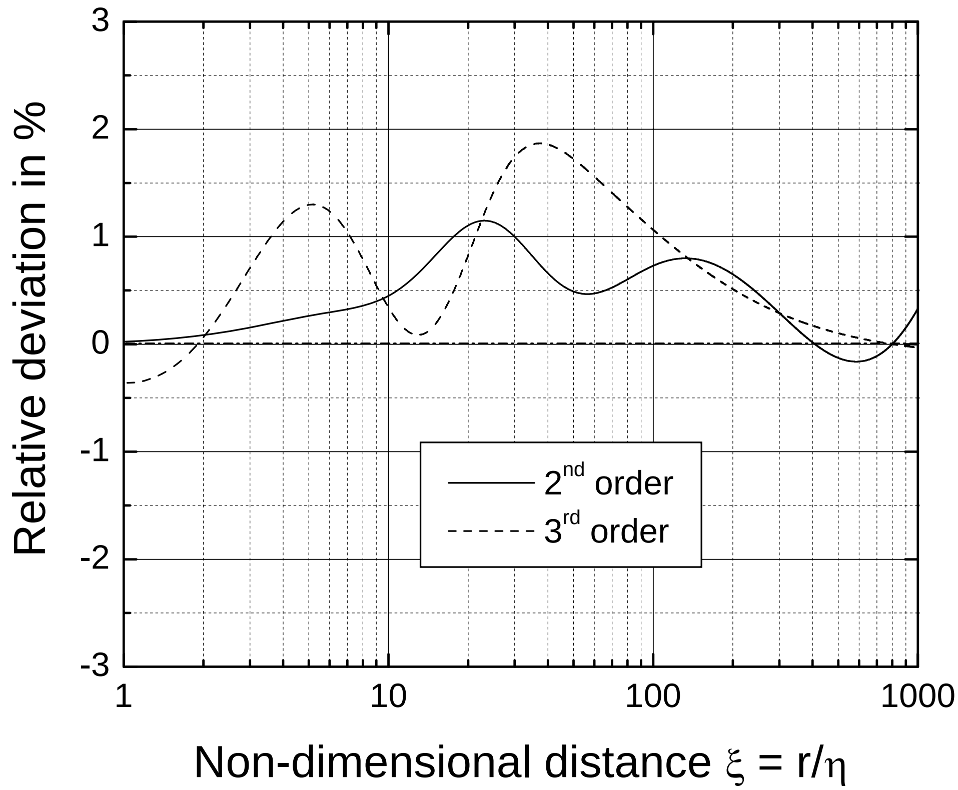

Figure 9: Deviation of the numerical solution of Eq. (3.42) from the interpolation formula (3.52) expressed in percent, when the skewness is fitted by the exponential function (3.53).

4. Structure functions for the temperature field

The structure function of the temperature field is considered as in example for a field of a passive scalar. Analogous to Kolmogorov's (1941a) similarity hypotheses for the turbulence structure of the flow field in the inertial range and the far dissipation range as well as the intermediate transition range under locally isotropic conditions at sufficiently large Reynolds numbers, the structure function of the temperature field can be determined using the similarity hypothesis $F\left(\mathrm{D}_{\mathrm{mT}}(\mathrm{r}), \overline{\mathrm{N}}/\bar{\varepsilon}, \bar{\varepsilon}, \mathrm{r}, \nu\right) = 0$ (see Eq. (A1)). Here, T is the absolute temperature,

$$\mathrm{D}_{\mathrm{mT}}(\mathrm{r}) = \mathrm{D}_{\mathrm{mT}}(\mathrm{M}, \mathrm{M}') = \overline{(\mathrm{T}(\mathrm{M}') - \mathrm{T}(\mathrm{M}))^{\mathrm{m}}} \tag{4.1}$$

is the structure function for the temperature field, $\overline{\mathrm{N}} = \overline{\alpha_{\mathrm{T}}\ (\nabla \mathrm{T})^2}$ (e.g., Obukhov, 1949; Lumley and Panofsky, 1964) is the mean dissipation rate for the temperature field describing the dissipation of temperature fluctuation, and α_{T} the thermal diffusivity. This mean dissipation rate is customarily considered to be independent of r. All other symbols have the same meaning like in chapter 3. The quantity $\overline{\mathrm{N}}/\bar{\varepsilon}$ relates the dissipation of the temperature field to the dissipation of kinetic energy. It may be called the dissipation ratio. As both quantities are considered to be independent of r, the dissipation ratio may be considered as independent of r, too. The similarity hypothesis for the temperature field implies that the number of the dimensional quantities is $\mathrm{k} = 5$. The table of fundamental dimensional quantities is given by

	$\mathrm{D}_{\mathrm{mT}}(\mathrm{r})$	$\overline{\mathrm{N}}/\bar{\varepsilon}$	$\bar{\varepsilon}$	r	ν
Length	0	-2	2	1	2
Time	0	2	-3	0	-1
Temperature	m	2	0	0	0

that leads to the dimensional matrix

$$\mathbf{G} = \begin{Bmatrix} 0 & -2 & 2 & 1 & 2 \\ 0 & 2 & -3 & 0 & -1 \\ \mathrm{m} & 2 & 0 & 0 & 0 \end{Bmatrix} \quad . \tag{4.2}$$

Thus, the system of linear equations reads (see Eq. (A9)),

$$\begin{Bmatrix} 0 & -2 & 2 & 1 & 2 \\ 0 & 2 & -3 & 0 & -1 \\ \mathrm{m} & 2 & 0 & 0 & 0 \end{Bmatrix} \begin{Bmatrix} \alpha_{1,\mathrm{i}} \\ \alpha_{2,\mathrm{i}} \\ \alpha_{3,\mathrm{i}} \\ \alpha_{4,\mathrm{i}} \\ \alpha_{5,\mathrm{i}} \end{Bmatrix} = \{\mathbf{0}\} \quad \text{for } \mathrm{i} = 1, 2 \quad . \tag{4.3}$$

As the rank of the dimensional matrix is $R = 3$, we have $p = k - R = 2$ independent π numbers. They can be derived from

$$\left.\begin{array}{lllllll} & -2\,\alpha_{2,i} & +2\,\alpha_{3,i} & +\alpha_{4,i} & +2\,\alpha_{5,i} & = & 0 \\ & 2\,\alpha_{2,i} & -3\,\alpha_{3,i} & & -\alpha_{5,i} & = & 0 \\ m\,\alpha_{1,i} & +2\,\alpha_{2,i} & & & & = & 0 \end{array}\right\} \quad . \tag{4.4}$$

Choosing $\alpha_{1,1} = 1$, $\alpha_{4,1} = 0$, $\alpha_{1,2} = 0$, and $\alpha_{4,2} = 1$ yields $\alpha_{2,1} = -m/2$, $\alpha_{3,1} = -m/4$, $\alpha_{5,1} = -m/4$, $\alpha_{2,2} = 0$, $\alpha_{3,2} = 1/4$, and $\alpha_{5,2} = -3/4$. It follows (see Eq. (A2)):

$$\pi_1 = \left(D_{mT}(r)\right)^1 \left(\frac{\overline{N}}{\bar{\varepsilon}}\right)^{-\frac{m}{2}} \bar{\varepsilon}^{-\frac{m}{4}}\, r^0\, \nu^{-\frac{m}{4}} = D_{mT}(r)\left(\frac{\overline{N}}{\bar{\varepsilon}}\right)^{-\frac{m}{2}} \left(\bar{\varepsilon}\,\nu\right)^{-\frac{m}{4}} \tag{4.5}$$

and

$$\pi_2 = \left(D_{mT}(r)\right)^0 \left(\frac{\overline{N}}{\bar{\varepsilon}}\right)^0 \bar{\varepsilon}^{\frac{1}{4}}\, r^1\, \nu^{-\frac{3}{4}} = r\left(\frac{\bar{\varepsilon}}{\nu^3}\right)^{\frac{1}{4}} = \frac{r}{\eta} = \xi \quad . \tag{4.6}$$

According to Eq. (A3b), we may write

$$\pi_1 = D_{mT}(r)\left(\frac{\overline{N}}{\bar{\varepsilon}}\right)^{-\frac{m}{2}} \left(\bar{\varepsilon}\,\nu\right)^{-\frac{m}{4}} = \varphi_{mT}(\pi_2) = \varphi_{mT}(\xi) \tag{4.7}$$

or

$$D_{mT}(\xi) = \left(\frac{\overline{N}}{\bar{\varepsilon}}\right)^{\frac{m}{2}} \left(\bar{\varepsilon}\,\nu\right)^{\frac{m}{4}} \varphi_{mT}(\xi) \quad . \tag{4.8}$$

Since two π numbers occur, a universal function, $\varphi_{mT}(\xi)$, is established again. According to this equation, the second-order structure function reads

$$D_{2T}(\xi) = \frac{\overline{N}}{\bar{\varepsilon}} \left(\bar{\varepsilon}\,\nu\right)^{\frac{1}{2}} \varphi_{2T}(\xi) \quad . \tag{4.9}$$

The third-order structure function is given by

$$D_{3T}(\xi) = \left(\frac{\overline{N}}{\bar{\varepsilon}}\right)^{\frac{3}{2}} (\bar{\varepsilon}\,\nu)^{\frac{3}{4}}\, \varphi_{3T}(\xi) \quad . \tag{4.10}$$

4.1 The asymptotic solution for the inertial range

Analogous to Kolgomorov’s (1941a) second similarity hypothesis regarding the turbulence structure of the inertial range, the similarity hypothesis for the temperature field is formulated by $F\left(D_{mT}(r), \overline{N}/\bar{\varepsilon}, \bar{\varepsilon}, r\right) = 0$. Again, viscosity plays no role. This may be assumed when the condition $\eta << r << \Lambda$ is fulfilled. The solution for this similarity hypothesis reads (e.g., Lesieur, 1997)

$$D_{mT}(r) = \pi_1 \left(\frac{\overline{N}}{\bar{\varepsilon}}\right)^{\frac{m}{2}} \bar{\varepsilon}^{\frac{m}{3}}\, r^{\frac{m}{3}} \quad . \tag{4.11}$$

This equation can be derived analogous to the respective asymptotic solution for the flow field (see section 3.1). For $m = 2$, we again obtain the two-thirds law,

$$D_{2T}(r) = \pi_1 \frac{\overline{N}}{\bar{\varepsilon}} \bar{\varepsilon}^{\frac{2}{3}}\, r^{\frac{2}{3}} = \pi_1\, \overline{N}\, \bar{\varepsilon}^{-\frac{1}{3}}\, r^{\frac{2}{3}} \quad , \tag{4.12}$$

where $\pi_1 = \chi_{2T} \cong 3.2$ (Kaimal et al., 1972; Panofsky and Dutton, 1984; Kaimal and Finnigan, 1994) is also a non-dimensional universal constant. Equation (4.12) may also be written as

$$D_{2T}(r) = C_{2T}{}^2\, r^{\frac{2}{3}} \quad . \tag{4.13}$$

Here,

$$C_{2T}{}^2 = \chi_{2T}\, \overline{N}\, \bar{\varepsilon}^{-\frac{1}{3}} \tag{4.14}$$

is called the structure parameter for temperature (see also Eq. (1.4)) that is closely identified with the structure parameter of the refractive index, $C_n{}^2$, for acoustic and electromagnetic waves propagating to the atmosphere (e.g., Panofsky and Dutton, 1984; Kaimal and Finnigan, 1994). Note that this structure parameter is only defined for the inertial range, too. If we consider the universal function for the energy dissipation for the ASL again, (see Eq. (3.18)), the structure parameter can be related to the universal function $\Phi_\varepsilon(\zeta)$ by

$$C_{2T}{}^2 = \chi_{2T}\, u_*{}^2\, \frac{\overline{N}}{\bar{\varepsilon}} \left\{\frac{\Phi_\varepsilon(\zeta)}{\kappa\,(z-d)}\right\}^{\frac{2}{3}} \quad . \tag{4.15}$$

where, again, $\Phi_\varepsilon(\zeta)$ is given by Eq. (3.20).

According to Eq. (4.11), the asymptotic solution for the third-order moment reads

$$D_{3T}(r) = \pi_1 \left(\frac{\overline{N}}{\bar{\varepsilon}}\right)^{\frac{3}{2}} \bar{\varepsilon}\, r \quad . \tag{4.16}$$

Again, there is a linear dependency on r.

The asymptotic solution of the universal function, $\varphi_{mT}(\xi)$, for the inertial range, $\eta << r << \Lambda$, is given by

$$\varphi_{mT}(\xi) = \chi_{mT}\, \xi^{\frac{m}{3}} \quad . \tag{4.17}$$

The universal functions of second-order and third-orders read then

$$\varphi_{2T}(\xi) = \chi_{2T}\, \xi^{\frac{2}{3}} \tag{4.18}$$

and

$$\varphi_{3T}(\xi) = \chi_{3T}\, \xi \quad . \tag{4.19}$$

Obviously, it differs from that of the flow field (see Eq. (3.23)) only by a number. In the case of the second-order universal functions the number amounts to $n \cong 1.69$.

If we use the similarity hypothesis $F\left(D_{d,2T}(r), \overline{N}/\bar{\varepsilon}, \bar{\varepsilon}, r\right) = 0$ for the third moment, the asymptotic solution for the inertial range ($\eta << r << \Lambda$) will read (see Eq. (C12))

$$D_{d,2T}(r) = \pi_1 \frac{\overline{N}}{\bar{\varepsilon}} \bar{\varepsilon}\, r = \pi_1\, \overline{N}\, r \quad , \tag{4.20}$$

where, according to Yaglom (1949), the π number is identified with $\pi_1 = -4/3$ (see also Chassaing et al., 2002). Using S_T yields then

$$D_{2T}(r) = \frac{D_{d,2T}(r)}{S_T\, \{D_{2d}(r)\}^{\frac{1}{2}}} = -\frac{4}{3} \frac{\overline{N}\, r}{S_T \left\{\chi_{2d}\, \bar{\varepsilon}^{\frac{2}{3}}\, r^{\frac{2}{3}}\right\}^{\frac{1}{2}}} = -\frac{4}{3} \frac{1}{S_T\, \chi_{2d}^{\frac{1}{2}}} \frac{\overline{N}}{\bar{\varepsilon}} \bar{\varepsilon}^{\frac{2}{3}}\, r^{\frac{2}{3}} \quad . \tag{4.21}$$

Therefore, the non-dimensional universal constant, χ_{2T}, that occurs in Eq. (4.12), can be related to the statistical properties S and S_T by (Yaglom, 1949)

$$\chi_{2T} = -\frac{4}{3}\frac{1}{S_T\,\chi_{2d}^{\frac{1}{2}}} = -\frac{4}{3}\frac{\left(-\frac{5}{4}S\right)^{\frac{1}{3}}}{S_T} \cong 3.2 \quad , \tag{4.22}$$

From this equation we can infer

$$S_T = -\frac{4}{9}\chi_{2d}^{-\frac{1}{2}} \cong -0.32 \quad . \tag{4.23}$$

However, as illustrated in Figure 10, also in the case of S_T there is a scatter, but not so large like in the case of the skewness (see Figure 2), where the results also suggest a dependency of S_T on the Reynolds number.

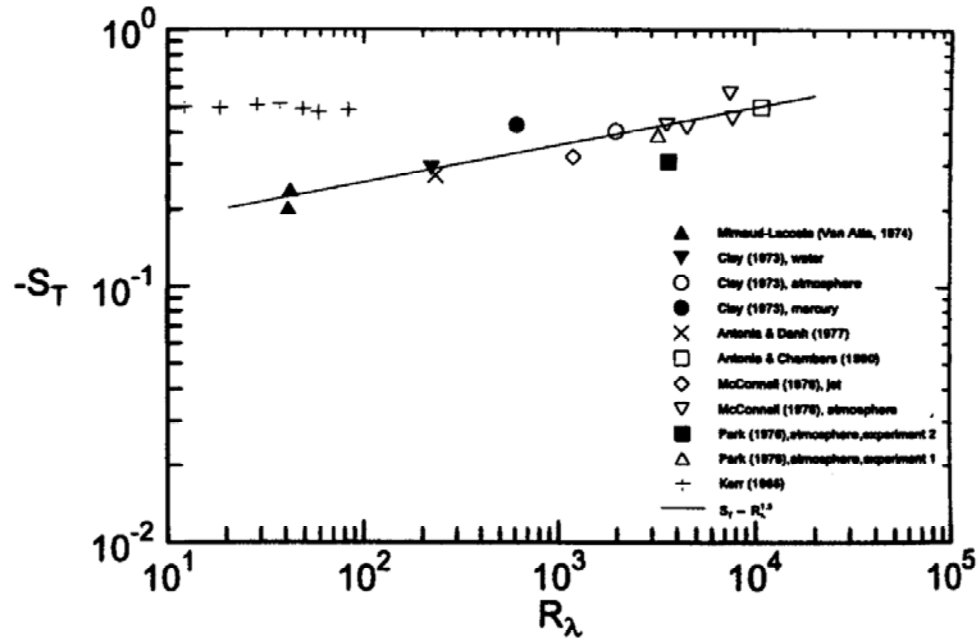

Figure 11: The Reynolds number (R_λ) variation of the mixed skewness ($-S_T$) (from Sreenivasan and Antonia, 1997).

4.2 The asymptotic solution for the far dissipation range

On the analogy of the derivation of the asymptotic solution for the turbulence structure of the far dissipation range ($r << \eta$) described in section 3.2, the similarity hypothesis for the temperature field may be written as $F\left(D_{mT}(r),\ \overline{N}/\bar{\varepsilon},\ \bar{\varepsilon}/\nu,\ r\right) = 0$. This similarity hypothesis leads to

$$D_{mT}(r) = \pi_1 \left(\frac{\overline{N}}{\bar{\varepsilon}}\right)^{\frac{m}{2}} \left(\frac{\bar{\varepsilon}}{\nu}\right)^{\frac{m}{2}} r^{m} \quad . \tag{4.24}$$

For $m = 2$, we again obtain the well-known r^2 law

$$D_{2T}(r) = \pi_1 \frac{\overline{N}}{\bar{\varepsilon}} \frac{\bar{\varepsilon}}{\nu} r^2 \quad , \tag{4.25}$$

where $\pi_1 = \delta_{2T} = \Pr/3$ (Obukhov, 1949; Yaglom, 1949) is also considered as a non-dimensional universal constant, where $\Pr = \nu/\alpha_T$ is the Prandtl number. The asymptotic solution for the third-order moment reads

$$D_{3T}(r) = \pi_1 \left(\frac{\overline{N}}{\bar{\varepsilon}}\right)^{\frac{3}{2}} \left(\frac{\bar{\varepsilon}}{\nu}\right)^{\frac{3}{2}} r^{3} \quad . \tag{4.26}$$

The asymptotic solution of the universal function, $\varphi_{mT}(\xi)$, for the far dissipation range, $r << \eta$, is then given by

$$\varphi_{mT}(\xi) = \delta_{mT}\ \xi^{m} \tag{4.27}$$

leading to

$$\varphi_{2T}(\xi) = \delta_{2T}\ \xi^{2} \tag{4.28}$$

and

$$\varphi_{3T}(\xi) = \delta_{3T}\ \xi^{3} \quad , \tag{4.29}$$

the universal functions of second-order and third-order, respectively. Again, these solutions differ from those of the flow field (see Eqs. (2.6), (3.36), and (3.37)) only by numbers. In the case of the second-order universal function it amounts to a value of $n \cong 3.55$. Note, however, that these numbers usually depend on the Prandtl number.

If we use the similarity hypothesis $F\left(D_{d,2T}(r), \overline{N}/\bar{\varepsilon}, \bar{\varepsilon}/\nu, r\right) = 0$, the asymptotic solution for the mixed structure function for the velocity and the temperature fields in the far dissipation range ($r << \eta$) will read (see Eq. (C16))

$$D_{d,2T}(r) = \pi_1 \frac{\overline{N}}{\bar{\varepsilon}} \left(\frac{\bar{\varepsilon}}{\nu}\right)^{\frac{3}{2}} r^{3} \tag{4.30}$$

leading to

$$\varphi_{d,2T}(\xi) = \pi_1\ \xi^{\,3} \quad , \tag{4.31}$$

where $\delta_{d,2T}$ is a non-dimensional universal constant, too. The expression for the mixed skewness in the far dissipation range reads then

$$S_T = \frac{\delta_{d,2T}}{\delta_{2d}^{\frac{1}{2}}\ \delta_{2T}} \quad , \tag{4.32}$$

i.e., also in the far dissipation range the mixed skewness is independent of r and ξ, respectively. In the case of a constant mixed skewness, Eq. (4.32) may be used to estimate $\delta_{d,2T}$. In doing so, one obtains

$$\delta_{d,2T} \cong -\frac{4}{27}\left(\frac{\delta_{2d}}{\chi_{2d}}\right)^{\frac{1}{2}} \Pr \quad . \tag{4.33}$$

4.3 The interpolation formula for the transition range

Since the asymptotic solutions of the universal function, $\varphi_{2T}(\xi)$, for the inertial range and the far dissipation range differs only by certain numbers from those of $\varphi_{2d}(\xi)$ the interpolation formula for the transition range should be similar to that illustrated Figure 3, as qualitatively sketched by Obukhov (1949). If we replace the third-order structure function in Yaglom's structure equation (2.16) by Eq. (2.19), we will obtain

$$2\,\nu\,\frac{dD_{2T}(r)}{dr} - S_T\,\{D_{2d}(r)\}^{\frac{1}{2}}\,D_{2T}(r) = \frac{4}{3}\,\overline{N}\,r \quad . \tag{4.34}$$

Employing Eqs. (2.5) and (4.9), Kolmogorov's local scale of turbulence and the Prandtl number yields then

$$\frac{d\varphi_{2T}(\xi)}{d\xi} - \Pr\frac{S_T}{2}\,\{\varphi_{2d}(\xi)\}^{\frac{1}{2}}\,\varphi_{2T}(\xi) = \frac{2}{3}\Pr\,\xi \quad . \tag{4.35}$$

It was shown that the mixed skewness, S_T, is independent of the distance r in both cases of asymptotic solutions. As in the case of the skewness, this finding may serve to consider S_T as independent of r for the whole transition range, even though a behavior as illustrated in Figure 8 for the skewness seems to be more likely. Customarily, Eq. (4.23) is used to replace S_T in Eq. (4.35) which leads to

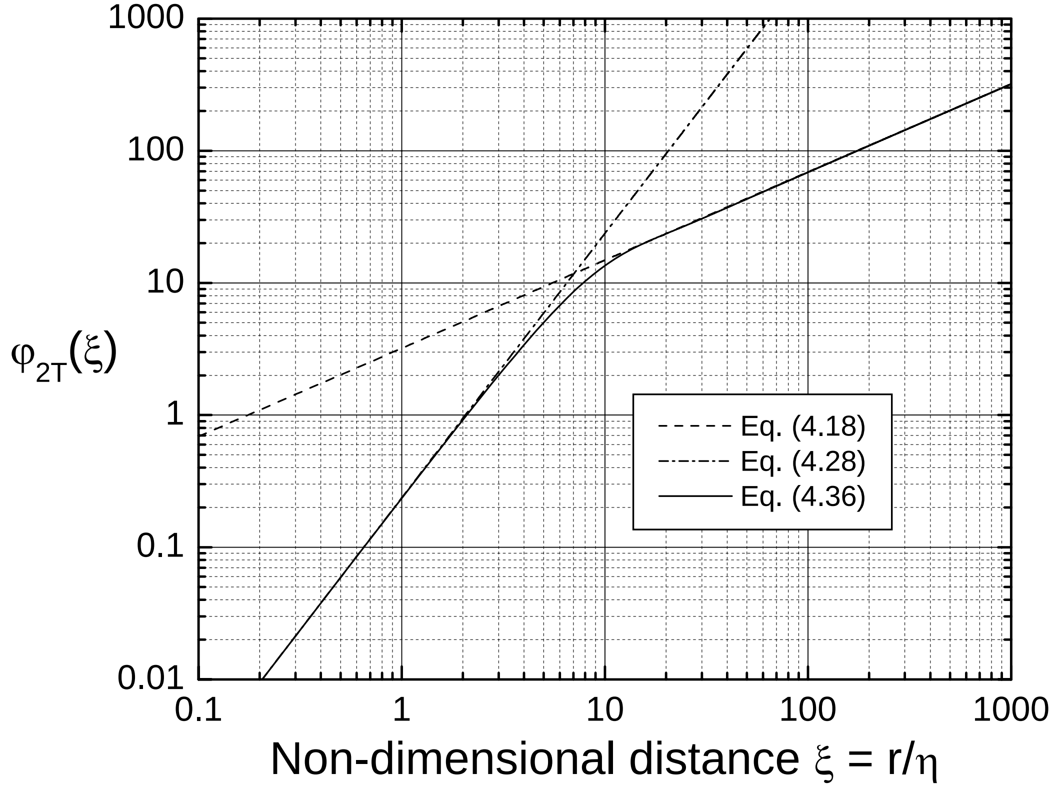

Figure 10: As in Figure 3, but for the universal function $\varphi_{2T}(\xi)$.

$$\frac{d\varphi_{2T}(\xi)}{d\xi} + \Pr \frac{2}{9} \chi_{2d}^{-\frac{1}{2}} \{\varphi_{2d}(\xi)\}^{\frac{1}{2}} \varphi_{2T}(\xi) = \frac{2}{3} \Pr \xi \quad . \tag{4.36}$$

This equation has to be solved numerically, too. The universal function $\varphi_{2d}(\xi)$ has to be provided by the simultaneous solution of Eq. (3.42) or by the interpolation formula (3.43). The result of the numerical solution of Eq. (4.36) is illustrated in Figure 10. As expected, it matches the asymptotic solutions of both the inertial range and the far dissipation range.

5. Similarity laws for the atmospheric surface layer

In studies on the turbulent structure of the ASL, the most important prerequisite states that the magnitude of the Reynolds stress vector and the vertical components of the turbulent flux densities of sensible heat and water vapor (here designated as micrometeorological fluxes) are invariant with height. From a mathematical point of view, it can be expressed by $\partial F/\partial z = 0 \Leftrightarrow F = \mathrm{const}(z)$, where, again, z is the height, and F stands for the micrometeorological fluxes of momentum (i.e., the magnitude of the Reynolds stress vector),

$$\tau = |\boldsymbol{\tau}| = \left(\overline{\rho\, u''\, w''}^{\,2} + \overline{\rho\, v''\, w''}^{\,2} \right)^{\frac{1}{2}} = \text{const.} \quad , \tag{5.1}$$

sensible heat,

$$H = c_{p,0}\, \overline{\rho\, w''\, \Theta''} = \text{const.} \quad , \tag{5.2}$$

and water vapor,

$$W = \overline{\rho\, w''\, q''} = \text{const.} \quad , \tag{5.3}$$

respectively. Again, ρ is the air density, and $c_{p,0}$ is the specific heat at constant pressure for dry air. Furthermore, u'', v'', and w'' are the fluctuations of the wind vector components in x-, y- and z-direction, with respect to their mean values $\hat{u}$, $\hat{v}$, and $\hat{w}$, where in this contribution $\hat{v}$ is arbitrarily chosen as equal to zero. This choice can be justified by arranging the x-axis in such a manner that $\hat{v}$ vanishes. Moreover, Θ'' is the fluctuation of the potential temperature with respect to the mean potential temperature, $\hat{\Theta}$, and q'' is the fluctuation of the specific humidity, $q = \rho_w / \rho$, with respect to the mean specific humidity, $\hat{q}$, where ρ_w is the partial density of water vapor. As already mentioned in the introduction, Hesselberg's (1926) density-weighted average, $\hat{f} = \overline{\rho\, f} / \overline{\rho}$, is applied, where f represents u, v, w, Θ, as well as q, and the overbar ($\overline{}$) designates the Reynolds' mean customarily used.

Since these micrometeorological fluxes are considered to be proportional to the vertical gradients of the mean field quantities, i.e., $F \propto \partial \hat{f} / \partial z$, where $f = u$ for $F = \tau$, $f = \Theta$ for $F = H$, and $f = q$ for $F = W$, this height invariance serves to mediate simple connections of vertical profiles of the mean quantity $\hat{f}$ to the turbulent flux F under diabatic (i.e., stable and unstable) stratification. It generally demands that steady-state conditions and the condition of horizontally uniform fields of mean wind speed (i.e., $\hat{w} = 0$), mean temperature, and mean humidity are fulfilled. In addition, net source and sink effects owing to phase transition processes must not occur. Even though the condition of height invariance may customarily be fulfilled only in a micrometeorological sense (i.e., these micrometeorological fluxes may vary with height across the whole ASL, but not more than 10 percent of their values in the immediate vicinity of the surface), it serves as the basis for the so-called constant flux approximation on which micrometeorological scaling is based, namely (a) Monin-Obukhov scaling (Monin and Obukhov, 1954) for forced convective conditions and (b) Prandt-Obukhov-Priestley scaling (Prandtl, 1932; Obukhov, 1946; Priestley, 1959) for free convective conditions, respectively. This kind of scaling is used here to derive the asymptotic solutions for free convective conditions.

5.1 Complete similarity versus incomplete similarity

Let us consider the transfer of momentum (index m), sensible heat (index h) and water vapor (index q) across the ASL. The standard procedures of dimensional analysis give (Barenblatt, 1996)

$$\pi_m = \frac{z-d}{u_*}\frac{\partial \hat{u}}{\partial z} = \varphi_m\left(\zeta, \mathrm{Re}_l, \mathrm{Re}_*, \mathrm{Pr}, \mathrm{Sc}_q\right) \quad , \tag{5.4}$$

$$\pi_h = \frac{z-d}{\Theta_*}\frac{\partial \hat{\Theta}}{\partial z} = \varphi_h\left(\zeta, \mathrm{Re}_l, \mathrm{Re}_*, \mathrm{Pr}, \mathrm{Sc}_q\right) \tag{5.5}$$

and

$$\pi_q = \frac{z-d}{q_*}\frac{\partial \hat{q}}{\partial z} = \varphi_q\left(\zeta, \mathrm{Re}_l, \mathrm{Re}_*, \mathrm{Pr}, \mathrm{Sc}_q\right) \quad . \tag{5.6}$$

Here, $\zeta = (z-d)/L$ is, again, the Obukhov number, where d is the zero-plane displacement, and L is the Obukhov stability length given by Eq. (1.6), $\mathrm{Re}_l = u_*\, z/\nu$ is the local Reynolds number, $\mathrm{Re}_* = u_*\, z_R/\nu$ is the global Reynolds number, z_R is an external geometric length scale for the ASL, usually its thickness, and $\mathrm{Sc}_q = \nu / D_q$ is the Schmidt number for water vapor, where D_q is the corresponding molecular diffusivity. As pointed out by Barenblatt (1996), this kind of scaling is based on the assumption of complete similarity of the flow in both Reynolds numbers, i.e. the local one, Re_l, and the global one, Re_*. The plausibility of such an assumption and, consequently, of neglecting the dependence on Re_l and Re_* is usually argued on the basis of the very large values of both Reynolds numbers above the thin sublayer, a layer of a few millimeters thickness adjacent to the earth's surface usually called the viscous sublayer. The assumption of the existence of finite limits of the local stability functions $\varphi_m(\zeta)$ for momentum, $\varphi_h(\zeta)$ for sensible heat, and $\varphi_q(\zeta)$ for water vapor as both Reynolds numbers tend to infinity is accepted implicitly. If $\varphi_m(\zeta)$, $\varphi_h(\zeta)$, and $\varphi_q(\zeta)$ tend to finite limits as $\mathrm{Re}_l \to \infty$ and $\mathrm{Re}_* \to \infty$ in accordance with the assumption of complete similarity, then for sufficiently large Re_l and Re_* a universal similarity law, independent of both Reynolds numbers, must hold (Barenblatt, 1996); and the local stability functions for momentum, sensible heat and water vapor,

$$\pi_m = \frac{z-d}{u_*}\frac{\partial \hat{u}}{\partial z} = \varphi_m\left(\zeta, \mathrm{Pr}, \mathrm{Sc}_q\right) \quad , \tag{5.7}$$

$$\pi_h = \frac{z-d}{\Theta_*}\frac{\partial \hat{\Theta}}{\partial z} = \varphi_h\left(\zeta, \mathrm{Pr}, \mathrm{Sc}_q\right) \tag{5.8}$$

and

$$\pi_q = \frac{z-d}{q_*}\frac{\partial \hat{q}}{\partial z} = \varphi_q\left(\zeta, \mathrm{Pr}, \mathrm{Sc}_q\right) \tag{5.9}$$

may be considered as universal functions. These are usually called the Monin-Obukhov similarity laws (Monin and Obukhov, 1954; Barenblatt, 1996). Note that in the case of the fully turbulent ASL the dependence of these universal functions on both the Prandt number and the Schmidt number for water vapor is usually neglected.

It is known that already in the case of thermally neutral stratification one can detect a weak dependence of these universal functions on both Reynolds numbers. The weak dependence serves to introduce the assumption of incomplete similarity of the flow in the local Reynolds number, which is apparently not contradicted by experimental data on flows in smooth pipes, etc. (Barenblatt, 1996). Thus, Barenblatt and Monin (1976, 1979) made a similar assumption for thermally stratified flows in the ASL.

6. Monin-Obukhov scaling

6.1 Flux-gradient relationship for momentum

Considering the first similarity hypothesis of Monin and Obukhov (1954) that states that the vertical transfer of momentum across the ASL is only determined by $z-d$, L (given by Eq. (1.6)), u_*, and $\partial\hat{u}/\partial z$, the similarity hypothesis can be expressed by $F(z-d, L, u_*, \partial\hat{u}/\partial z) = 0$ (see Eq. (A1)). Obviously, the number of dimensional quantities is $k = 4$. This fact is true in all instances of flux-gradient relationships presented here. The dimensional matrix is given by

$$G = \begin{Bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{Bmatrix} \tag{6.1}$$

that can be derived from the table of fundamental dimensions,

	$z-d$	L	u_*	$\partial\hat{u}/\partial z$
Length	1	1	1	0
Time	0	0	-1	-1

In accord with Eq. (A9), the linear equation set is given by

$$\begin{Bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{Bmatrix} \begin{Bmatrix} \alpha_{1,i} \\ \alpha_{2,i} \\ \alpha_{3,i} \\ \alpha_{4,i} \end{Bmatrix} = \{0\} \quad \text{for } i = 1, 2 \quad . \tag{6.2}$$

Obviously, the rank of the dimensional matrix is $R = 2$, and we have $p = k - R = 2$ independent π numbers, i.e., a universal function is established. These facts are true in all instances of flux-gradient relationships presented here. These two π numbers can be derived from

$$\left.\begin{array}{lllll}\alpha_{1,i} & +\alpha_{2,i} & +\alpha_{3,i} & & = 0 \\ & & \alpha_{3,i} & +\alpha_{4,i} & = 0\end{array}\right\} \quad \text{for } i = 1, 2 \quad . \tag{6.3}$$

Choosing, for instance, $\alpha_{1,1} = 1$, $\alpha_{3,1} = -1$, $\alpha_{1,2} = 1$, and $\alpha_{2,2} = -1$ yields $\alpha_{2,1} = 0$, $\alpha_{4,1} = 1$, $\alpha_{3,2} = 0$, and $\alpha_{4,2} = 0$. According to Eq. (A2), the independent π numbers are given by

$$\pi_1 = (z-d)^1 \, L^0 \, u_*^{-1} \left(\frac{\partial \hat{u}}{\partial z}\right)^1 = \frac{z-d}{u_*} \frac{\partial \hat{u}}{\partial z} \tag{6.4}$$

and

$$\pi_2 = (z-d)^1 \, L^{-1} \, u_*^{0} \left(\frac{\partial \hat{u}}{\partial z}\right)^0 = \frac{z-d}{L} = \zeta \quad . \tag{6.5}$$

By considering Eq. (A3b), we may write (see also Eq. (5.7))

$$\pi_1 = \frac{z-d}{u_*} \frac{\partial \hat{u}}{\partial z} = \varphi_m(\pi_2) = \varphi_m(\zeta) \quad . \tag{6.6a}$$

Apparently, Eq. (6.6a) does not contain κ. This is quite reasonable because the von Kármán constant is a non-dimensional quantity. In the case of neutral stratification for which the similarity hypothesis $F(z-d, u_*, \partial\hat{u}/\partial z) = 0$ is suitable (the Obukhov stability length may be removed from the table of fundamental dimensions presented before), κ is related to a π number by $\pi_1 = \varphi_m(0) = \kappa^{-1}$ (e.g., Barenblatt, 1979, 1996). In the case of diabatic stratification, however, it cannot directly be addressed by a formalized procedure of non-dimensionalization. Following Monin and Obukhov (1954), we may put this constant into Eq. (6.6a), but mainly for historical reasons and convenience. It follows

$$\frac{\kappa(z-d)}{u_*} \frac{\partial \hat{u}}{\partial z} = \Phi_m(\zeta) \quad , \tag{6.6b}$$

where $\Phi_m(\zeta) = \kappa \, \varphi_m(\zeta)$ is the local similarity function (or universal function) for momentum (Monin and Obukhov, 1954; Lumley and Panofsky, 1964). Note that the quantity $l_P = \kappa(z-d)$ in Eq. (6.6b) is customarily designated as Prandtl's mixing length of the universal wall law for neutral stratification characterized by $\Phi_m(0) = 1$. This kind of stratification, however, can only be considered as a special condition either (a) during the transition from stable to unstable stratification and vice versa (when steady-state conditions, as required by the constant flux approximation, are not to be expected, e.g., Stearns (1970); Kramm (1989)) or (b) under the condition of strong wind shear when, in addition, diabatic effects are of minor importance.

In all instances with more than one π number, a universal function cannot be quantified by the dimensional π-invariants analysis. As mentioned before, the local similarity function for momentum must be equal to unity when $\zeta = 0$. But for diabatic conditions as originally investigated by Monin and Obukhov (1954), it has to be determined empirically or/and theoretically.

Integrating Eq. (6.6b) over the height interval $[z_r, z_R]$, where z_r and z_R are the lower and upper boundaries of the fully turbulent part of the ASL, respectively, yields

$$\left.\begin{aligned}\hat{u}(z_R) - \hat{u}(z_r) &= \frac{u_*}{\kappa}\int_{z_r}^{z_R} \frac{\Phi_m((z-d)/L)}{z-d}\,dz = \frac{u_*}{\kappa}\int_{z_r}^{z_R} \frac{1 - 1 + \Phi_m((z-d)/L)}{z-d}\,dz \\ &= \frac{u_*}{\kappa}\left(\int_{z_r}^{z_R} \frac{1}{z-d}\,dz - \int_{z_r}^{z_R} \frac{1-\Phi_m((z-d)/L)}{z-d}\,dz\right) = \frac{u_*}{\kappa}\left(\ln\frac{z_R - d}{z_r - d} - \Psi_m(\zeta_R, \zeta_r)\right)\end{aligned}\right\} \tag{6.7}$$

with

$$\Psi_m(\zeta_R, \zeta_r) = \int_{z_r}^{z_R} \frac{1-\Phi_m((z-d)/L)}{z-d}\,dz = \int_{\zeta_r}^{\zeta_R} \frac{1-\Phi_m(\zeta)}{\zeta}\,d\zeta \tag{6.8}$$

that is called the integral similarity function for momentum.

Some historical notes seem to be indispensable. Equation (6.7) was derived first by Panofsky (1963) to obtain the so-called logarithmic wind profile when for neutral stratification $\Phi_m(\zeta)$ approaches to unity leading to $\Psi_m(\zeta_R, \zeta_r) = 0$. The integral similarity function, $\Psi_m(\zeta_R, \zeta_r)$, is defined by Eq. (6.8), i.e., this definition is independent of the shape of $\Phi_m(\zeta)$. Furthermore, Eqs. (6.7) and (6.8) underline that $\Psi_m(\zeta_R, \zeta_r)$ is not the integral over $\Phi_m(\zeta)$ as recently stated in the literature. Moreover, Eq. (6.8) was first solved (a) by Monin and Obukhov (1954) for stable stratification (and weakly unstable stratification) assuming the linear function $\Phi_m(\zeta) = 1 + \beta\,\zeta$, later experimentally proved by Čalikov (1968), Zilitinkevič and Čalikov (1968), Businger et al. (1971) and others for the stability range $0 \le \zeta < 1$, to obtain the so-called logarithmic-linear wind profile, and (b) by Paulson (1970) using the so-called Businger-Dyer-Pandolfo relationship for unstable stratification $\Phi_m(\zeta) = (1 - \gamma\,\zeta)^{-1/4}$ (Dyer, unpublished, see Businger, 1988; Businger, 1966; Pandolfo, 1966), later experimentally proved by Dyer and Hicks (1970), Businger et al. (1971) and others, where their results mainly cover the stability range $-2 \le \zeta < 0$ (e.g., Panofsky and Dutton, 1984; Sorbjan, 1989). The quantities β and γ are considered as universal constants, even though the results from a number of serious field experiments show a considerable scatter (see, e.g., Table VI and VII in Högström, 1988, and Figures 12 and 13) and may also depend on ζ (Fortak, 1969; Herbert and Panhans, 1979; Panhans and Herbert, 1979; Kramm et al., 1996).

Paulson (1970) also solved Eq. (6.8) by using the conventional O'KEYPS equation[1], $\Phi_m{}^4(\zeta) - \delta\,\zeta\,\Phi_m{}^3(\zeta) = 1$, that may be considered as an interpolation formula for the range between forced (neutral stratification) and free convection (e.g., Lumley and Panofsky, 1964), where δ is another universal constant. It indicates a $\Phi_m(\zeta) \cong (-\delta\,\zeta)^{-1/3}$ behavior for large negative Obukhov numbers, for which $\Phi_m(\zeta) << -\delta\,\zeta$ becomes valid. Local similarity functions of the form $\Phi_m(\zeta) = (1 - \gamma\,\zeta)^{-1/3}$ as found, for instance, by Carl et al. (1973) as well as Gavrilov and Petrov (1981) for unstable stratification, $-10 \le \zeta < 0$, reflect the same asymptotic behavior, but they disagree with that of the Businger-Dyer-Pandolfo relationship. The O'KEYPS equation with $\delta = 9$ is experimentally proved for the range $-2 \le \zeta < 0$ (Businger et al., 1971); Panofsky and Dutton (1984), however, recommended: $\delta = 15$. From a physical point of view the O'KEYPS formula seems to be more preferable than the Businger-Dyer-Pandolfo relationship because this formula can simply be related to the balance equation of the turbulent kinetic energy (Fortak, 1969; Herbert and Panhans, 1979; Panhans and Herbert, 1979; Kramm et al., 1996).

Introducing the $\Phi_m(\zeta)$-function for stable stratification and the Businger-Dyer-Pandolfo relationship into Eq. (6.8) yields (Kramm and Herbert, 1984; Kramm, 1989)

$$\Psi_m(\zeta_R, \zeta_r) = \begin{cases} -\beta\,(\zeta_R - \zeta_r) & \text{for } L > 0 \\ 0 & \text{for } L \to \infty \\ 2\ln\dfrac{1 + y_R}{1 + y_r} + \ln\dfrac{1 + y_R{}^2}{1 + y_r{}^2} - 2\arctan\dfrac{y_R - y_r}{1 + y_R\,y_r} & \text{for } L < 0 \end{cases} \tag{6.9}$$

with $y_{r,R} = \Phi_m{}^{-1}(\zeta_{r,R}) = (1 - \gamma\,\zeta_{r,R})^{1/4}$, the reciprocal expressions of the local similarity functions in the unstable case at the two heights z_r and z_R. These equations completely agree with those of Paulson (1970) when y_r approaches to unity while $\zeta_r \to 0$. Introducing the conventional O'KEYPS formula into Eq. (6.8) provides

$$\left.\begin{aligned} \Psi_m(\zeta_R, \zeta_r) = \Phi_m(\zeta_r) - \Phi_m(\zeta_R) + 2\ln\frac{1 + \Phi_m(\zeta_R)}{1 + \Phi_m(\zeta_r)} + \ln\frac{1 + \Phi_m{}^2(\zeta_R)}{1 + \Phi_m{}^2(\zeta_r)} \\ + 2\arctan\frac{\Phi_m(\zeta_R) - \Phi_m(\zeta_r)}{1 + \Phi_m(\zeta_R)\,\Phi_m(\zeta_r)} - 3\ln\frac{\Phi_m(\zeta_R)}{\Phi_m(\zeta_r)} \end{aligned}\right\} \text{for } L \le 0 \tag{6.10}$$

[1] O'KEYPS stands for the initials of various authors who proposed this formula (see Obukhov, 1946; Kazansky and Monin, 1956; Ellison, 1957; Yamamoto, 1959; Panofsky, 1961; Sellers, 1962).

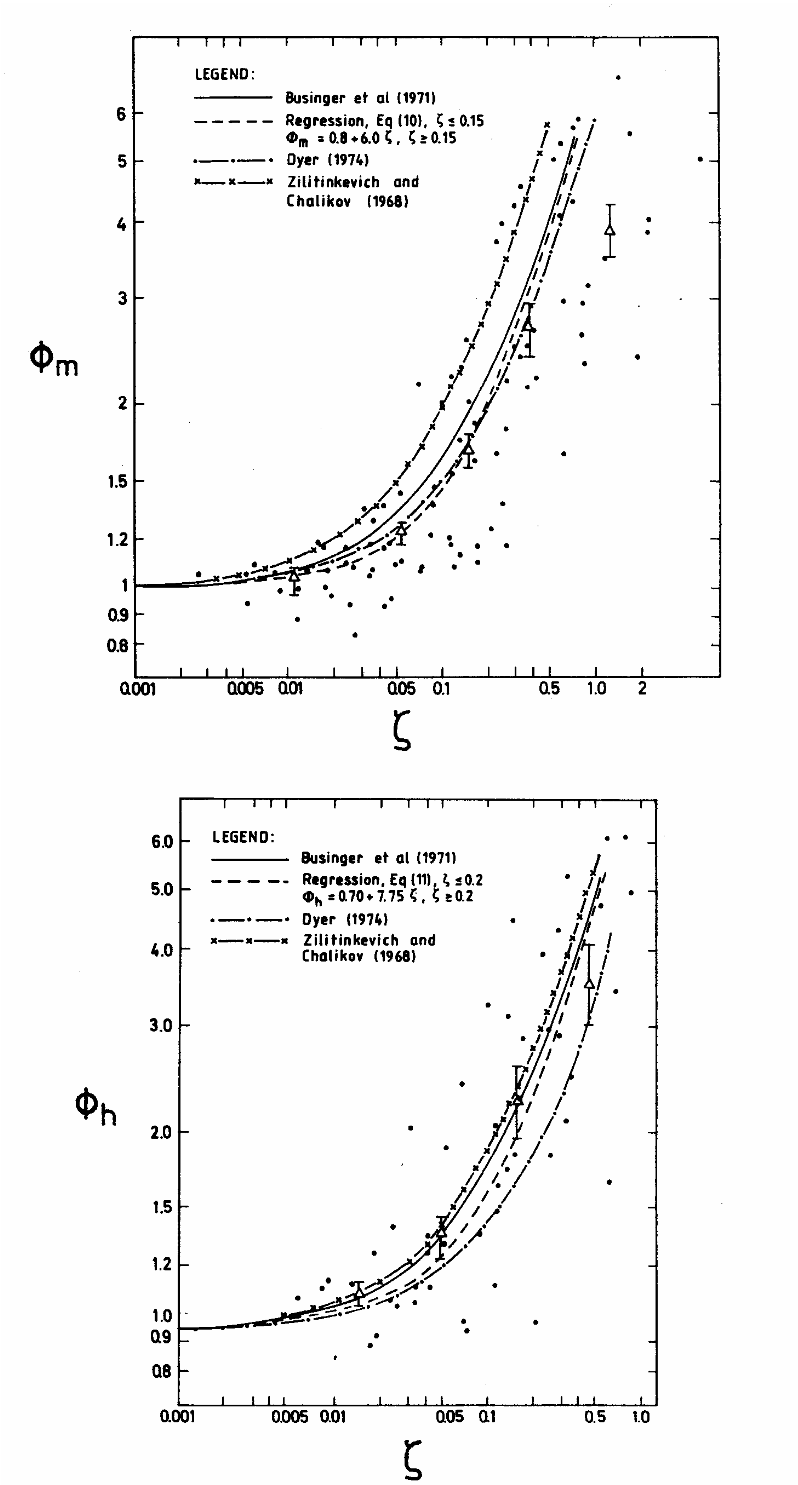

Figure 12: Variation of the non-dimensional wind speed and temperature gradients with the Obukhov number for stable stratification (from Högström, 1988).

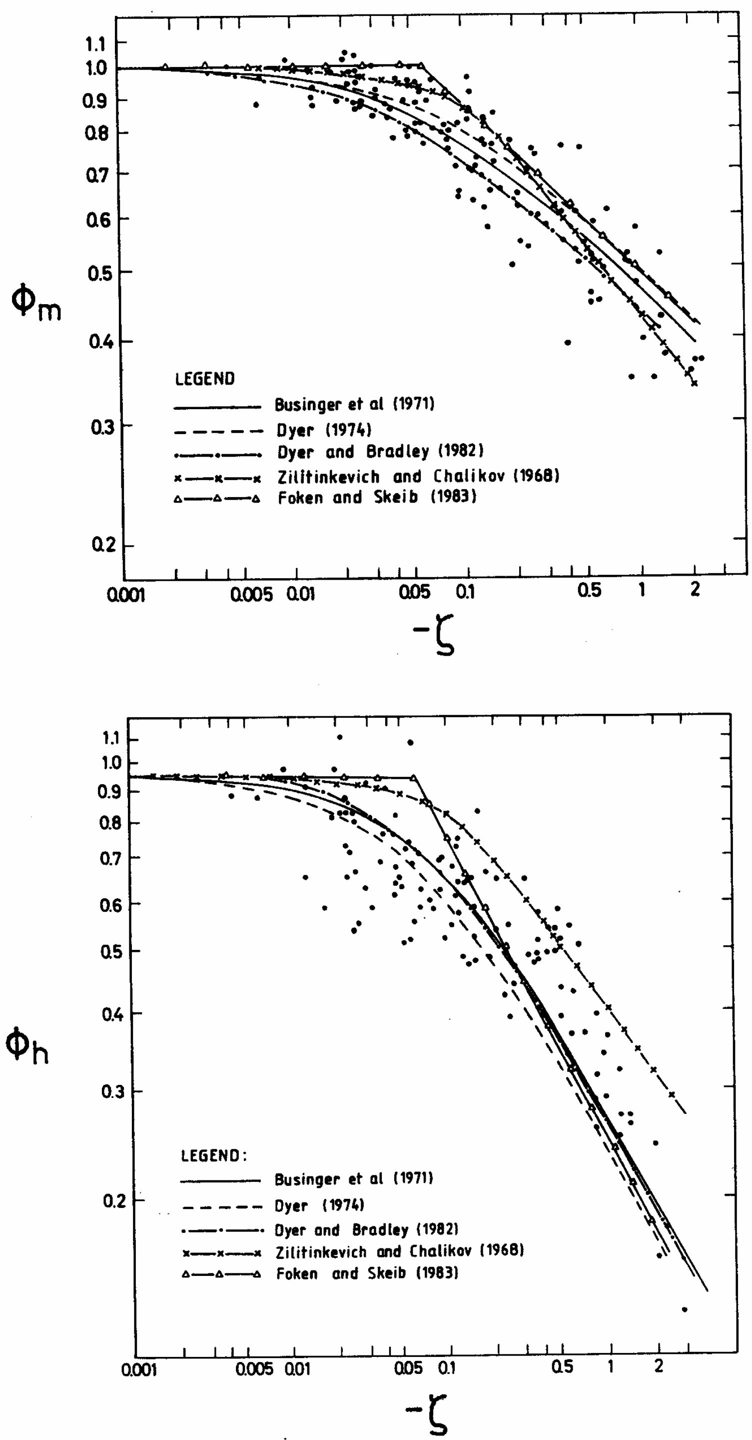

Figure 13: As in Figure 12, but for unstable stratification (from Högström, 1988).

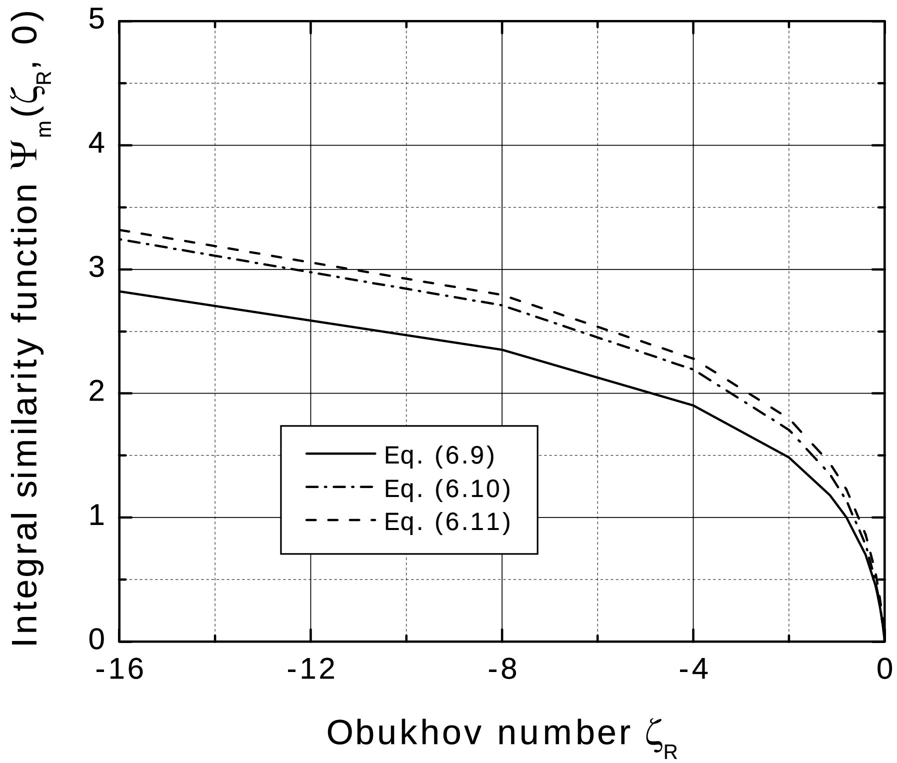

Figure 14: The integral similarity function $\Psi_m(\zeta_R, 0)$ for momentum obtained from Eqs. (6.9), (6.10), and (6.11) plotted against the Obukhov number $\zeta_R = z_R / L$.

with $\Phi_m(\zeta_{r,R})$ derived from the O'KEYPS formula. Obviously, the O'KEYPS solution (6.10) is more bulky than that obtained with the Businger-Dyer-Pandolfo relationship. This might be the reason why the latter is more widely used, even if the former has a stronger physical background. For $\zeta_r \to 0$, we have $\Phi_m(\zeta_r) \to 1$, and, hence, Eq. (6.10) approaches to Paulson's (1970) O'KEYPS-solution. Introducing $\Phi_m(\zeta) = (1 - \gamma\,\zeta)^{-1/3}$ into Eq. (6.8) provides

$$\Psi_m(\zeta_R, \zeta_r) = \frac{3}{2} \ln \frac{y_R^{\,2} + y_R + 1}{y_r^{\,2} + y_r + 1} - \sqrt{3}\,\arctan \frac{x_R - x_r}{1 + x_R\, x_r} \quad \text{for } L \le 0 \tag{6.11}$$

with $y_{r,R} = \Phi_m^{-1}(\zeta_{r,R}) = (1 - \gamma\,\zeta_{r,R})^{1/3}$, the reciprocal expressions of the local similarity functions in the unstable case at the two heights z_r and z_R, and $x_{r,R} = (2\,y_{r,R} + 1)/\sqrt{3}$. It approaches to Lettau's (1979) solution when ζ_r tends to zero. Equations (6.9), (6.10), and (6.11) are illustrated in Figure 14. As expected, Eqs. (6.10) and (6.11) only differ hardly when ζ tends to large negative Obukhov numbers that represent free convective conditions. Simultaneously, the difference between Eq. (6.9) and the other two equations grows continuously.

6.2 Flux-gradient relationship for sensible heat

The second similarity hypothesis of Monin and Obukhov (1954) states that the vertical transfer of sensible heat across the ASL is only determined by $z-d$, L, Θ_*, and $\partial\hat{\Theta}/\partial z$, expressed by $F\left(z-d, L, \Theta_*, \partial\hat{\Theta}/\partial z\right)=0$. The dimensional matrix is given by

$$G=\begin{Bmatrix}1 & 1 & 0 & -1\\ 0 & 0 & 1 & 1\end{Bmatrix} \quad , \tag{6.12}$$

according to the table of fundamental dimensions

	$z-d$	L	Θ_*	$\partial\hat{\Theta}/\partial z$
Length	1	1	0	-1
Temperature	0	0	1	1

Thus, the linear equation set is given by (see Eq. (A9))

$$\begin{Bmatrix}1 & 1 & 0 & -1\\ 0 & 0 & 1 & 1\end{Bmatrix}\begin{Bmatrix}\alpha_{1,i}\\ \alpha_{2,i}\\ \alpha_{3,i}\\ \alpha_{4,i}\end{Bmatrix}=\{0\} \quad \text{for } i=1,2 \quad . \tag{6.13}$$

The two corresponding π numbers can be derived from

$$\left.\begin{array}{lllllll}\alpha_{1,i} & +\alpha_{2,i} & & -\alpha_{4,i} & = & 0\\ & & \alpha_{3,i} & +\alpha_{4,i} & = & 0\end{array}\right\} \quad \text{for } i=1,2 \quad . \tag{6.14}$$

Taking $\alpha_{1,1}=1$, $\alpha_{3,1}=-1$, $\alpha_{1,2}=1$, and $\alpha_{2,2}=-1$ into account, we obtain again: $\alpha_{2,1}=0$, $\alpha_{4,1}=1$, $\alpha_{3,2}=0$, and $\alpha_{4,2}=0$. These powers lead to the following π numbers (see Eq. (A2)):

$$\pi_1=(z-d)^1\, L^0\, \Theta_*^{-1}\left(\frac{\partial\hat{\Theta}}{\partial z}\right)^1=\frac{z-d}{\Theta_*}\frac{\partial\hat{\Theta}}{\partial z} \tag{6.15}$$

and

$$\pi_2=(z-d)^1\, L^{-1}\, \Theta_*^{0}\left(\frac{\partial\hat{\Theta}}{\partial z}\right)^0=\frac{z-d}{L}=\zeta \quad . \tag{6.16}$$

According to Eq. (A3b), we have

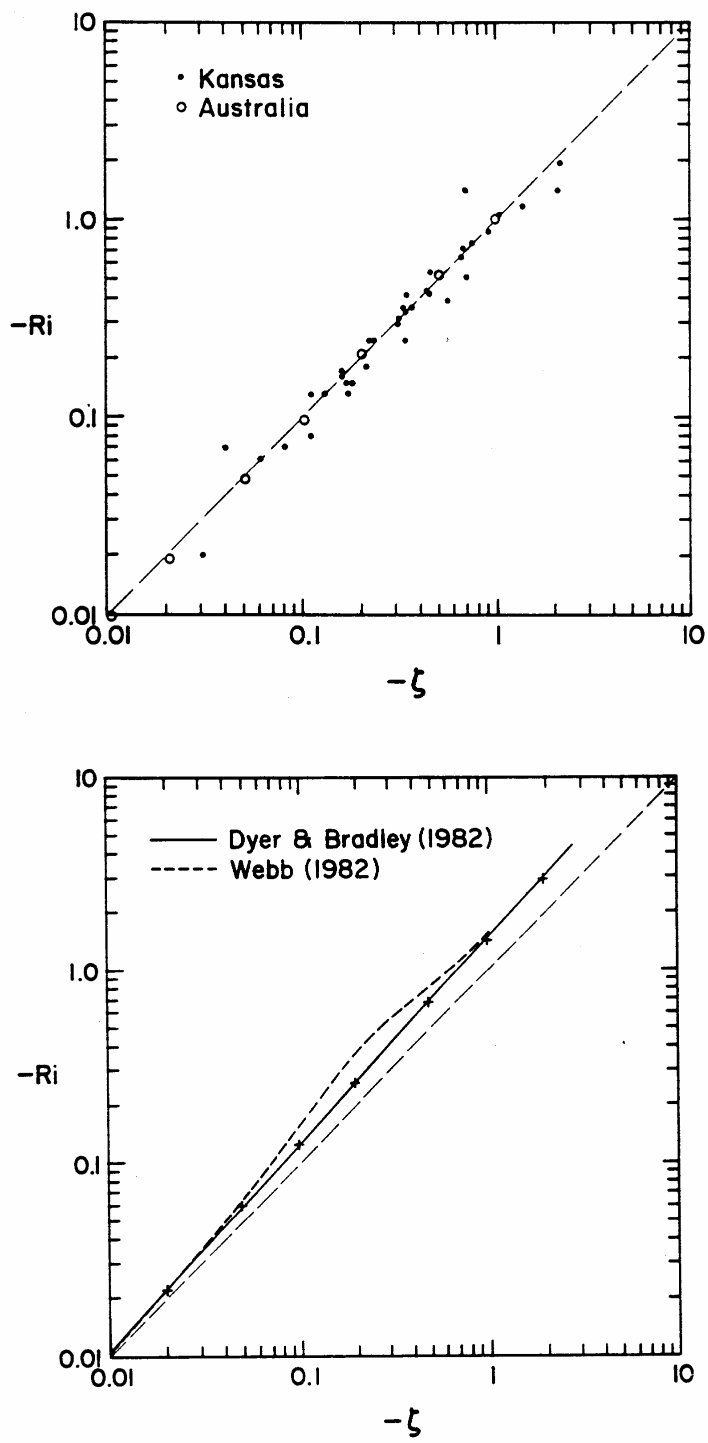

Figure 15: Gradient Richardson number, Ri, versus Obukhov number, ζ, for various field experiments (from Businger, 1988).

$$\pi_1 = \frac{z-d}{\Theta_*}\frac{\partial\hat{\Theta}}{\partial z} = \varphi_h(\pi_2) = \varphi_h(\zeta) \quad . \tag{6.17a}$$

As before, the von Kármán constant may be put into Eq. (6.17a) for historical reasons and convenience. In doing so, we have

$$\frac{\kappa(z-d)}{\Theta_*}\frac{\partial\hat{\Theta}}{\partial z} = \Phi_h(\zeta) \quad , \tag{6.17b}$$

where $\Phi_h(\zeta) = \kappa\,\varphi_h(\zeta)$ is the local similarity function (or universal function) for sensible heat (Monin and Obukhov, 1954). Sometimes, an additional factor α_h is introduced into Eq. (6.17b),

$$\frac{\alpha_h\ \kappa(z-d)}{\Theta_*}\frac{\partial\hat{\Theta}}{\partial z} = \alpha_h\ \Phi_h(\zeta) = \Phi_h{}^*(\zeta) \tag{6.17c}$$

with $\Phi_h{}^*(\zeta) = \alpha_h\ \Phi_h(\zeta)$, to address that the turbulent Prandtl number, $\Pr_t = \Phi_h(\zeta)/\Phi_m(\zeta)$, differ from unity for neutral stratification (e.g., Foken, 2003). Such an additional factor might be used for convenience. But it cannot be justified on the basis of any kind of dimensional analysis. Under neutral condition with respect to dry air, as considered by Businger et al. (1971), the similarity hypothesis mentioned before is not fulfilled because the vertical component of the temperature gradient, $Q_4 = \partial\hat{\Theta}/\partial z$, is equal to zero, and, hence, Eq. (6.17a, b, c) becomes unpredictable. Consequently, Eq. (6.17c) is not further considered here.

Equation (6.17b) can be integrated over the height interval $[z_r, z_R]$ in a similar manner like in the case of momentum (see Eqs. (6.7) and (6.8)). One obtains

$$\hat{\Theta}(z_R) - \hat{\Theta}(z_r) = \frac{\Theta_*}{\kappa}\left(\ln\frac{z_R-d}{z_r-d} - \Psi_h(\zeta_R, \zeta_r)\right) \tag{6.18}$$

with

$$\Psi_h(\zeta_R, \zeta_r) = \int_{\zeta_r}^{\zeta_R} \frac{1-\Phi_h(\zeta)}{\zeta}\, d\zeta \tag{6.19}$$

that is called the integral similarity function for sensible heat. As in the case of momentum, this definition is independent of the shape of the respective local similarity function (here, $\Phi_h(\zeta)$). Equation (6.19) was first solved (a) by Monin and Obukhov (1954) by assuming the same linear function for sensible heat as for momentum, $\Phi_h(\zeta) = \Phi_m(\zeta)$, for stable stratification (and weakly unstable stratification), later experimentally proved, for instance, by Zilitinkevič and Čalikov (1968), getting the logarithmic-linear temperature profile and (b) by Paulson (1970) using the relationship for unstable stratification $\Phi_h(\zeta) = \Phi_m{}^2(\zeta)$, suggested by Businger (1966) and Pandolfo (1966), later experimentally proved, for instance, by Dyer and Hicks (1970) for the stability range $-1 \le \zeta < 0$. This relationship leads to $\mathrm{Ri} = \zeta$, where Ri is the gradient-Richardson number. Dyer and Bradley (1982) as well as Webb (1982), however, pointed out that small deviations from this identity might occur (see Figures 15 and 28).

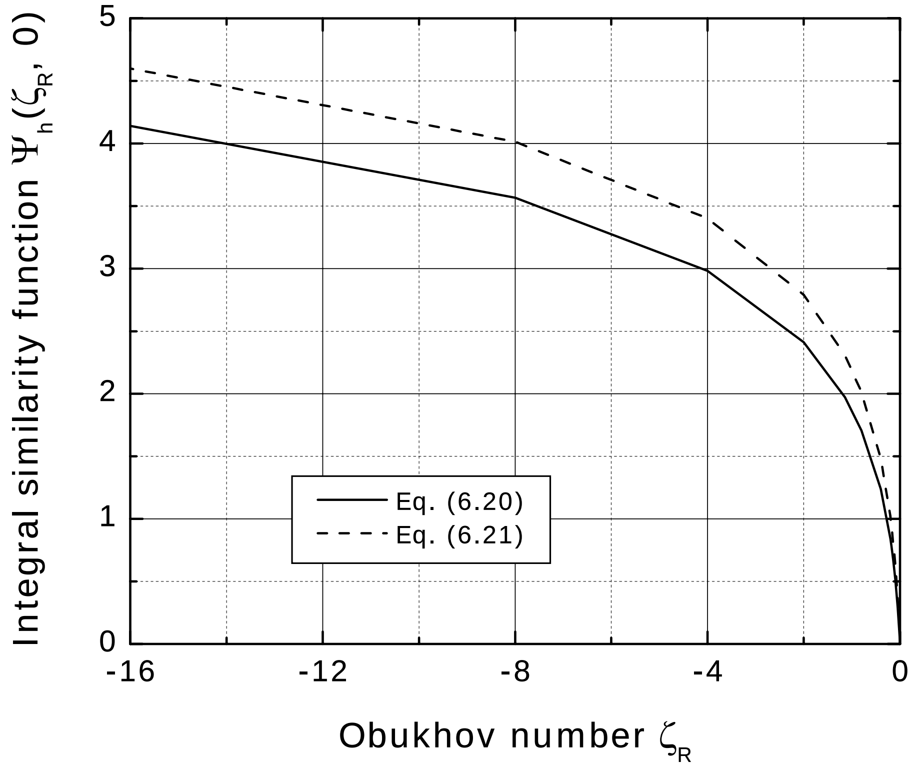

Figure 16: The integral similarity function $\Psi_h(\zeta_R, 0)$ for sensible heat obtained from Eqs. (6.20) and (6.21) plotted against the Obukhov number $\zeta_R = z_R / L$.

Introducing the $\Phi_m(\zeta)$-function for stable stratification and the Businger-Dyer-Pandolfo relationship into Eq. (6.19) yields (Kramm and Herbert, 1984; Kramm, 1989)

$$\Psi_h(\zeta_R, \zeta_r) = \begin{cases} \Psi_m(\zeta_R, \zeta_r) & \text{for } L > 0 \\ 0 & \text{for } L \to \infty \\ 2 \ln \dfrac{1 + y_R{}^2}{1 + y_r{}^2} & \text{for } L < 0 \end{cases} \tag{6.20}$$