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# ON THE TRANSFER OF MOMENTUM, SENSIBLE HEAT AND MATTER ACROSS THE INTERFACIAL SUBLAYER OVER AERODYNAMICALLY SMOOTH SURFACES

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**Abstract.** The transfer of momentum, sensible heat and matter across the interfacial sublayer over aerodynamically smooth surfaces is described by using a new interpolation formula for the normalized eddy diffusivity, $K_m/\nu$, that exactly fulfills Reichardt's (1950, 1951) criteria regarding the asymptotic behavior of $K_m/\nu$ in the viscous sublayer and in the inertial sublayer when an exponent of $n = 3$ is chosen (reference case). Results provided by numerical integration techniques using this interpolation formula are plotted against the flow velocity data of Nikuradse (1932), Reichardt (1950), and Laufer (1954) and the temperature data of Deissler and Eian (1952) for the purpose of evaluation. It is shown that the numerical results well agree with these laboratory data. In addition, the results of this reference case are compared with those obtained with the new interpolation formula and an exponent of $n = 4$ and the $K_m/\nu$-approaches of Reichardt (1950, 1951), van Driest (1956), and Spalding (1961, 1964). Even though the interpolation formula of the reference case differs from that of Spalding (1961, 1964), especially in the viscous sublayer, the results obtained with both interpolation formulae only differ hardly (smaller than 2 percent). Notably larger differences occur in the transition layer and in the inertial sublayer when $n = 4$ is chosen or when the interpolation formulae of Reichardt (1950, 1951) and van Driest (1956) are applied.

**1. Introduction.** We know that the transfer of momentum, sensible heat and gaseous matter across the thin interfacial sublayer (abbreviated by ISL) in the immediate vicinity of any surface is strongly controlled and limited by molecular transfer properties (e.g., Owen and Thomson 1963, Chamberlain 1968, Kramm et al. 1992, 1996, 2002). As illustrated in Figure 1, the ISL consists of the viscous sublayer, the buffer or transition layer, and the inertial sublayer approaching the fully turbulent region (also called logarithmic boundary layer) where molecular effects become negligible (e.g., Kestin and Richardson 1963, Schlichting 1965, Tennekes and Lumley 1978).

Assuming an incompressible fluid, the transfer of momentum, sensible heat and gaseous matter across the ISL over aerodynamically smooth surfaces of solid and liquid material can be expressed by (e.g., Schlichting 1965. Rotta 1972, Tennekes and Lumley 1978, Kramm et al. 1995)

$$\tau = \bar{\rho} u_*^2 = \bar{\rho}\left(\nu \frac{\partial \bar{u}}{\partial z} - \overline{u'w'}\right) \tag{1.1}$$

and

$$F_i = \lambda \bar{\rho}\left(D_i \frac{\partial \bar{\chi}_i}{\partial z} - \overline{w'\chi_i'}\right) \tag{1.2}$$

respectively. Here, $\tau$ is the magnitude of the total friction vector, $\boldsymbol{\tau}$, parallel to the surface, $\rho$ is the air density, $u_*$ is the friction velocity defined by $u_* = \sqrt{\tau/\bar{\rho}}$, $\nu$ is the kinematic viscosity, $u'$ is the fluctuation of the velocity component $u$ parallel to the surface, and $w'$ is the fluctuation of the velocity component $w$ perpendicular to the surface ($z$-direction). We assume that the mean horizontal flow vector, $\bar{\mathbf{v}}_H = \bar{u}\mathbf{i}$, and the total friction stress vector, $\boldsymbol{\tau}$, are parallel, where $\mathbf{i}$ is the unit vector in $x$-direction, i.e., the transfer of momentum is considered as a two-dimensional problem. Furthermore, $F_i$ is the total flux density (simply denoted here as a flux) of a scalar entity ($i$ may stand for sensible heat and gaseous matter, respectively) toward or from that surface, and $\chi_i'$ is the fluctuation of the corresponding scalar quantity like the absolute temperature, $\chi_i = T$, or the mass fraction, $\chi_i = \rho_i/\rho$, where $\rho_i$ is the partial density. Moreover, $D_i$ represents the molecular diffusivity of the $i$-th gaseous entity in air and the thermal diffusivity, $\alpha$, respectively, $\lambda$ is equal to unity in the case of such gaseous constituents, and $\lambda = c_p$ stands for sensible heat, where $c_p$ is the specific heat of air at constant pressure. Note that an overbar $(\bar{\ })$ denotes a mean value according to Reynolds' averaging calculus, and a prime $(')$ the deviation from that. Furthermore, $\tau$ and, hence, $u_*$ as well as $F_i$ are considered as independent of $z$.

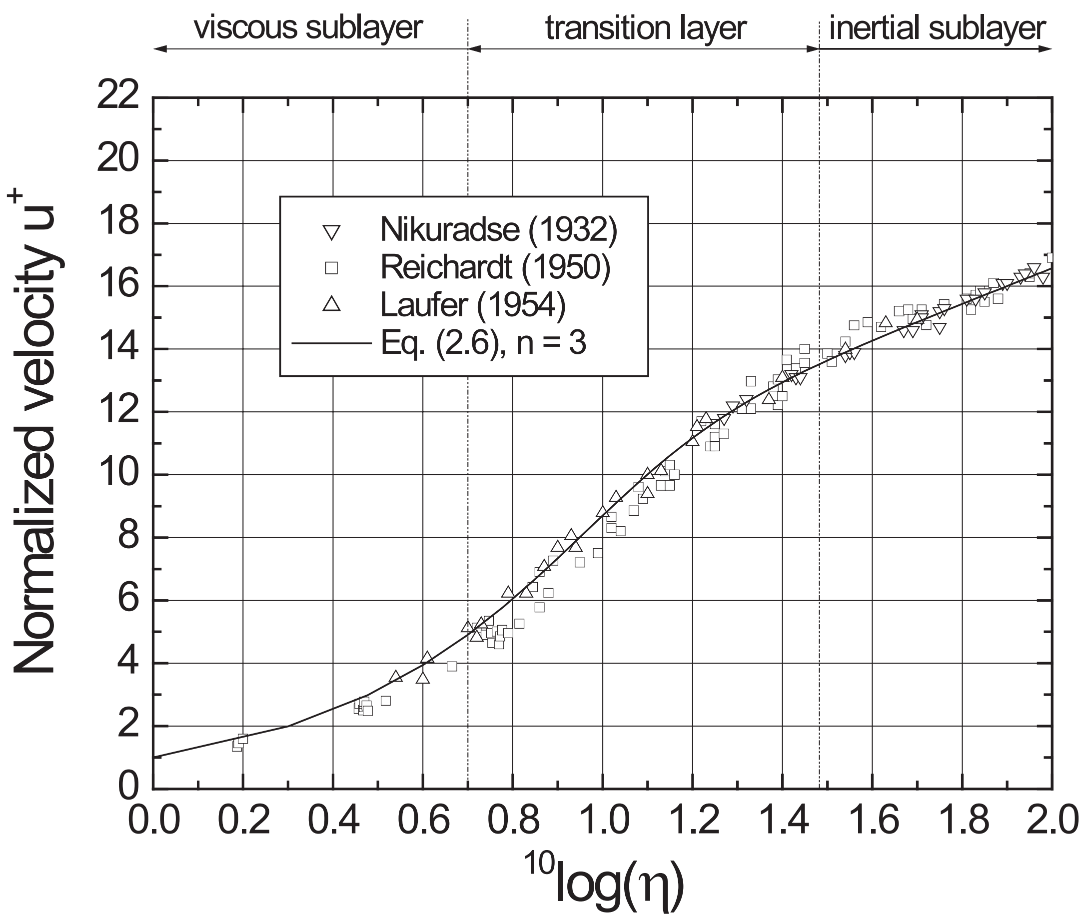


**Figure 1:** Normalized velocity profiles for the ISL over aerodynamically smooth surfaces calculated on the basis of Eq. (1.9), where the interpolation formula (2.6) with $n = 3$ is considered, and compared with the laboratory data of Nikuradse (1932), Reichardt (1950), and Laufer (1954). This velocity profile is called the reference velocity profile.

If we assume that the covariance terms in Eqs. (1.1) and (1.2) can be parameterized by considering first-order closure principles (flux-gradient relationships), we will obtain

$$\tau = \bar{\rho} u_*^2 = \bar{\rho}\nu \left(1 + \frac{K_m}{\nu}\right) \frac{\partial \bar{u}}{\partial z} = \bar{\rho} u_* \left(1 + \frac{K_m}{\nu}\right) \frac{\partial \bar{u}}{\partial \eta} \tag{1.3}$$

and

$$F_i = -\lambda \bar{\rho} D_i \left(1 + \frac{K_i}{D_i}\right) \frac{\partial \bar{\chi}_i}{\partial z} = -\lambda \bar{\rho} \frac{u_*}{X} \left(1 + \frac{X}{X_t} \frac{K_m}{\nu}\right) \frac{\partial \bar{\chi}_i}{\partial \eta} \tag{1.4}$$

where $X$ and $X_t$ are defined by

$$X = \begin{cases} Pr & \text{for sensible heat} \\ Sc_i & \text{for gaseous matter} \end{cases} \tag{1.5}$$

and

$$X_t = \begin{cases} Pr_t & \text{for sensible heat} \\ Sc_{t,i} & \text{for gaseous matter} \end{cases} \tag{1.6}$$

respectively. Here, $Pr = \nu/\alpha$ is the Prandtl number, $Pr_t = K_m/K_h$ is the turbulent Prandtl number, $Sc_i = \nu/D_i$ is the Schmidt number for the $i$-th gaseous entity, $Sc_{t,i} = K_m/K_i$ is the corresponding turbulent Schmidt number, $\eta = u_* z/\nu$ is the so-called roughness Reynolds number, and $K_m$, $K_h$, and $K_i$ are the eddy diffusivities for momentum, sensible heat and the $i$-th gaseous entity, respectively. The sum $\nu + K_m$ is customarily called the effective viscosity, and the sum $D_i + K_i$ is usually named the effective diffusivity. Generally, $Pr_t$ and $Sc_{t,i}$ have to be presumed as functions of $\eta$.

Integrating Eqs. (1.3) and (1.4) over the height interval $[0, z_r]$ of the ISL, where $z_r$ is its outer edge, yields then

$$\tau = \bar{\rho} u_*^2 = \bar{\rho} u_* \left(\frac{\xi_d}{2}\right)^{\frac{1}{2}} (\overline{u_r} - \overline{u_s}) = \text{const.} \tag{1.7}$$

and

$$F_i = -\lambda \bar{\rho} u_* \chi_{*,i} = -\lambda \bar{\rho} u_* B_i (\overline{\chi_{r,i}} - \overline{\chi_{s,i}}) = \text{const.} \tag{1.8}$$

Here, $\overline{u_r}$ and $\overline{u_s}$ are the mean values of the flow velocity at the outer edge of the ISL (subscript $r$), and at the surface (subscript $s$), where in the case of rigid walls, as considered here, the latter is equal to zero, $\overline{\chi_{r,i}}$ and $\overline{\chi_{s,i}}$ are the corresponding mean values of the scalar entity, $\xi_d = 2(u_*/\overline{u_r})^2$ is the local drag coefficient, and $B_i$ is the sublayer-Stanton number. Note that in the case of matter $B_i$ is also called the sublayer-Dalton number.

According to Eqs. (1.7) and (1.8), the key quantities that characterize the transfer ability of the ISL are the local drag coefficient and the sublayer-Stanton number. The latter can be considered as a measure of the difference in the corresponding transfers of momentum, sensible heat and gaseous matter towards and from aerodynamically rough surfaces (e.g., Dipprey and Sabersky 1963, Owen and Thomson 1963, Chamberlain 1968, Yaglom and Kader 1974) and aerodynamically smooth surfaces (e.g., Pohlhausen 1921, Brutsaert 1975, 1982, Kramm et al. 1992, 1996, 2002), respectively. As argued, for instance, by Chamberlain (1968), this difference results from the facts that (a) momentum is transmitted to the surface by skin friction and by form drag, and only the former has an analogy in mass and heat transfer, and (b) the molecular diffusion coefficient, $D_i$, of the $i$-th gaseous entity and the thermal diffusivity, $\alpha$, are not numerically equal to the analogous kinematic viscosity, $\nu$.

Surfaces will be considered as aerodynamically smooth if the characteristic height, $z_e$, of the roughness elements is smaller than the thickness of the viscous sublayer (i.e., $z_e < 5\nu/u_*$, see Nikuradse 1932). The influence of the roughness elements on the distribution of the mean flow velocity and, hence, on the transfer of sensible heat and matter cannot be detected (e.g., Rotta 1972). In such a case, $\xi_d$ and $B_i$ are given by (see also Kramm and Dlugi 1994, Kramm et al. 1995, 2002)

$$\left(\frac{\xi_d}{2}\right)^{-\frac{1}{2}} = \int_0^{\eta_r} \frac{d\eta}{1 + K_m/\nu} = \frac{\overline{u_r}}{u_*} = u_r^+ \tag{1.9}$$

and

$$B_i^{-1} = X \int_0^{\eta_r} \frac{d\eta}{1 + \frac{X}{X_t} K_m/\nu} = \frac{\overline{\chi_{r,i}} - \overline{\chi_{s,i}}}{\chi_{*,i}} \tag{1.10}$$

where $\eta_r = u_* z_r/\nu$ is the upper integration limit related to the thickness of the ISL. Note that for $X_t = X = 1$, the quantity $B_i^{-1}$ is equal to $(\xi_d/2)^{-1/2}$ and, hence, equal to $u_r^+$.

Obviously, the key quantities for properly solving Eqs. (1.9) and (1.10) are $K_m/\nu$ and $X_t$ being presumed that they generally depend on $\eta$. If these dependencies are known, Eqs. (1.9) and (1.10) can be solved either elementary or numerically, as done, for instance, by Kestin and Richardson (1963), Rotta (1964), Kramm et al. (1992, 1995, 1996, 2002) and Muschinski (1992).

In the following, the transfer of momentum and sensible heat across the ISL over aerodynamically smooth surfaces under forced-convective conditions will be quantified by numerical (Romberg) integration techniques using a new interpolation formula for the normalized eddy diffusivity, $K_m/\nu$, that exactly fulfills Reichardt's (1950, 1951) criteria regarding the asymptotic behavior of $K_m/\nu$ in the viscous sublayer and in the inertial sublayer when an exponent of $n = 3$ is chosen. The results of this numerical integration, denoted as the reference case, are compared with those obtained with the new interpolation formula and $n = 4$. In addition, they are also compared with those derived by Kramm et al. (2002) on the basis of the interpolation formulae of Reichardt (1950, 1951), van Driest (1956), and Spalding (1961, 1964). As compared with the $K_m/\nu$-approaches of Elser (1949), Deissler (1955), Rannie (1956), and Sheppard's (1958) as well as Roth's (1972) modified Heisenberg model for the spectral energy transfer in the equilibrium range under locally isotropic conditions, the interpolation formulae of Reichardt (1950, 1951), van Driest (1956), and Spalding (1961, 1964) provided the best results when plotted against laboratory data of flow velocity (Nikuradse 1932, Reichardt 1950, Laufer, 1954) and temperature (Deissler and Eian 1952). Kramm et al. (2002) considered not only a turbulent Prandtl number of $Pr_t = 1$, but also an average value of $Pr_t = 0.78$ (e.g., Reichardt 1950, Kestin and Richardson 1963, Schlichting 1965) and the empirical sigmoidal distribution,

$$Pr_t = \frac{0.649 - 0.952}{1 + \exp\left(\frac{\frac{r}{R} - 0.438}{0.162}\right)} + 0.952 \tag{1.11}$$

that is based on Ludwieg's (1956) measurements of the variation with the normalized radius $r/R$ ($r$ is the radial distance from the centre of the pipe, $R$ is the pipe radius) of the turbulent Prandtl number for air flowing in a pipe. His results indicate that $Pr_t$ varies smoothly and continuously from a value nearly 0.94 close to the pipe wall to a value of about 0.67 at the centre of the pipe, where a possible dependence on the Mach number cannot be not detected. According to Kestin and Richardson (1963) and Schlichting (1965), Ludwieg's (1956) results are closest to being correct. Kramm et al. (2002), however, showed that only in the case of Deissler's (1955) interpolation formula the turbulent Prandtl number given by Eq. (1.11) provides the best results. Unfortunately, the convergence behavior of this $K_m/\nu$-approach was insufficient during the numerical solution of Eq. (1.10) for $\eta > 690$ (see Kramm et al. 2002). Because of this lack, Deissler's (1955) interpolation formula is not further discussed, and, therefore, $Pr_t = 1$ is only considered in this study.

**2. Various $K_m/\nu$-approaches for the ISL.** As mentioned before, the normalized eddy diffusivity, $K_m/\nu$, is one of the two key quantities required by Eqs. (1.9) and (1.10). The interpolation formulae for $K_m/\nu$ of Reichardt (1950, 1951), van Driest (1956), and Spalding (1961, 1964) describe the transition range from a purely laminar transport to a fully turbulent transport where molecular effects become negligible (see also Figures 1 and 4). It is similar to the so-called film model, but avoids (a) the unrealistic notion of a purely molecular layer near a fully turbulent layer, and (b) the necessity to determine its thickness (Hasse and Liss 1980). The behavior of $K_m/\nu$ in the ISL when $\eta$ approaches to zero is described by exponential functions. Since, however, the turbulent velocity components have to satisfy the equation of continuity, Reichardt (1950, 1951) argued that the derivations $\partial(K_m/\nu)/\partial\eta$ and $\partial^2(K_m/\nu)/\partial\eta^2$ must vanish at the surface and that in the immediate vicinity of this surface the normalized eddy diffusivity should vary with the third power of $\eta$ (see also Elrod 1957, Hinze 1959, Monin and Yaglom 1971, and Appendix A). For large values of $\eta$, however, $K_m/\nu$ should linearly depend on $\eta$ so that the asymptotic relation for neutral stratification, $K_m/\nu = \kappa\eta$, on which the logarithmic velocity profile, the turbulent approximation of Prandtl and von Kármán for large Reynolds numbers (e.g., Schlichting 1965, Monin and Yaglom 1971, Rotta 1972, Tennekes and Lumley 1978),

$$u^+ = \frac{\bar{u}(\eta)}{u_*} = \frac{1}{\kappa}\ln(\eta) + 5.5 \tag{2.1}$$

is based. Here, $\kappa \cong 0.4$ is the von Kármán constant. Reichardt's (1950, 1951) also argued that the interpolation formula

$$\frac{K_m}{\nu} = \kappa(\eta - \eta_D \tanh\frac{\eta}{\eta_D}) \tag{2.2}$$

fulfills these asymptotic requirements. This, however, is not exactly true. The $K_m/\nu$-behavior for larger values of $\eta$ is notably affected by the constant $\eta_D$ which has to be determined empirically. This fact and its $\eta^3$-behavior for very small values of $\eta$ can be explained as follows: If we approximate the function $\tanh(\eta/\eta_D)$ by a Maclaurin series, we will obtain

$$\tanh\frac{\eta}{\eta_D} = \frac{\eta}{\eta_D} - \frac{1}{3}\left(\frac{\eta}{\eta_D}\right)^3 + \frac{2}{15}\left(\frac{\eta}{\eta_D}\right)^5 - \frac{17}{315}\left(\frac{\eta}{\eta_D}\right)^7 + \cdots \tag{2.3}$$

that converges for all values $|\eta/\eta_D| < \pi/2$: For small values of $\eta$, it is sufficient to consider the first two elements of this Maclaurin series. Thus, we have

$$\frac{K_m}{\nu} \cong \kappa \left( \eta - \eta_D \left( \frac{\eta}{\eta_D} - \frac{1}{3} \left( \frac{\eta}{\eta_D} \right)^3 \right) \right) = \frac{\kappa}{3\eta_D^2} \eta^3 \tag{2.4}$$

For increasing values of $\eta$ the influence of the hyperbolic tangent decreases rapidly because $\tanh(\eta/\eta_D)$ tends to unity when $\eta/\eta_D$ grows. In such a case we have

$$\frac{K_m}{\nu} = \kappa(\eta - \eta_D) \tag{2.5}$$

i.e., Reichardt's (1950, 1951) interpolation formula does not exactly converge to the asymptotic relation for neutral stratification, $K_m/\nu = \kappa\eta$. The amount of the deviation from this relation depends on the constant $\eta_D$. Based on the 131 data sets of Nikuradse (1932), Reichardt (1950), and Laufer (1954) shown in Figure 1 and $\kappa = 0.4$, Kramm et al. (2002) re-determined this constant by least squares fitting and obtained $\eta_D \cong 11.01$. Thus, for values of $\eta \leq 100$, the constant $\eta_D$ is not negligible as often assumed.

In contrast to Reichardt's (1950, 1951) $K_m/\nu$-approach, the interpolation formula

$$\frac{K_m}{\nu} = \frac{\eta_D \eta^n}{\left(1 + \left(\frac{\eta_D}{\kappa}\right)^{\frac{n}{n-1}} \eta^n\right)^{\frac{n-1}{n}}} \tag{2.6}$$

exactly fulfills Reichardt's criteria when $n = 3$ is chosen. For very small values of $\eta$ leading to $(\eta_D/\kappa)^{3/2}\eta^3 \ll 1$, we obtain $K_m/\nu = \eta_D\eta^3$; for large values of $\eta$ we have $(\eta_D/\kappa)^{3/2}\eta^3 \gg 1$, and, hence, $K_m/\nu = \kappa\eta$. The constant $\eta_D \cong 7.35 \cdot 10^{-4}$ was also determined by least squares fitting, as described before. Note that the interpolation formula (2.6) with $n = 3$ has the similar structure than that proposed by Kramm (2004) for the third-order structure function of the velocity field under isotropic conditions and sufficiently large Reynolds numbers which can be considered as an extension of the corresponding second-order structure function suggested by Batchelor (1951).

On the contrary, as illustrated in Figure 2, the interpolation formulae of van Driest (1956),

$$\frac{K_m}{\nu} = \frac{1}{2} \left( \left\{ 1 + 4 \left\{ \kappa\eta \left[ 1 - \exp\left( -\frac{\eta}{\eta_D} \right) \right] \right\}^2 \right\}^{1/2} - 1 \right) \tag{2.7}$$

with $\eta_D = 26.44$ (obtained by least squares fitting as described above), and Spalding (1961, 1964),

$$\frac{K_m}{\nu} = \kappa \exp(-\kappa\eta_D) \left\{ \exp(\kappa u^+) - \sum_{i=0}^{3} \frac{(\kappa u^+)^i}{i!} \right\} \tag{2.8}$$

with $\eta_D = 5.13$ (also obtained by least squares fitting), show an $\eta^4$-behavior. These approaches seem to be in contradiction to Reichardt's (1950, 1951) criteria, but he conceded that in the case of a perfect flow a $K_m/\nu$-behavior with a power larger than 3 might be possible.

If we choose $n = 4$, Eq. (2.6) will provide for very small values of $\eta$ (i.e., $(\eta_D/\kappa)^{4/3}\eta^4 \ll 1$) $K_m/\nu = \eta_D\eta^4$ with $\eta_D \cong 6.35 \cdot 10^{-5}$ (also obtained by least squares fitting) and for large values of

$\eta$ (i.e., $(\eta_D/\kappa)^{4/3}\eta^4 \gg 1$) $K_m/\nu = \kappa\eta$. Thus, compared with the $K_m/\nu$ approaches of Reichardt (1950, 1951), van Driest (1956), and Spalding (1961, 1964), Eq. (2.6) may be considered as a more general interpolation formula for the normalized eddy diffusivity.

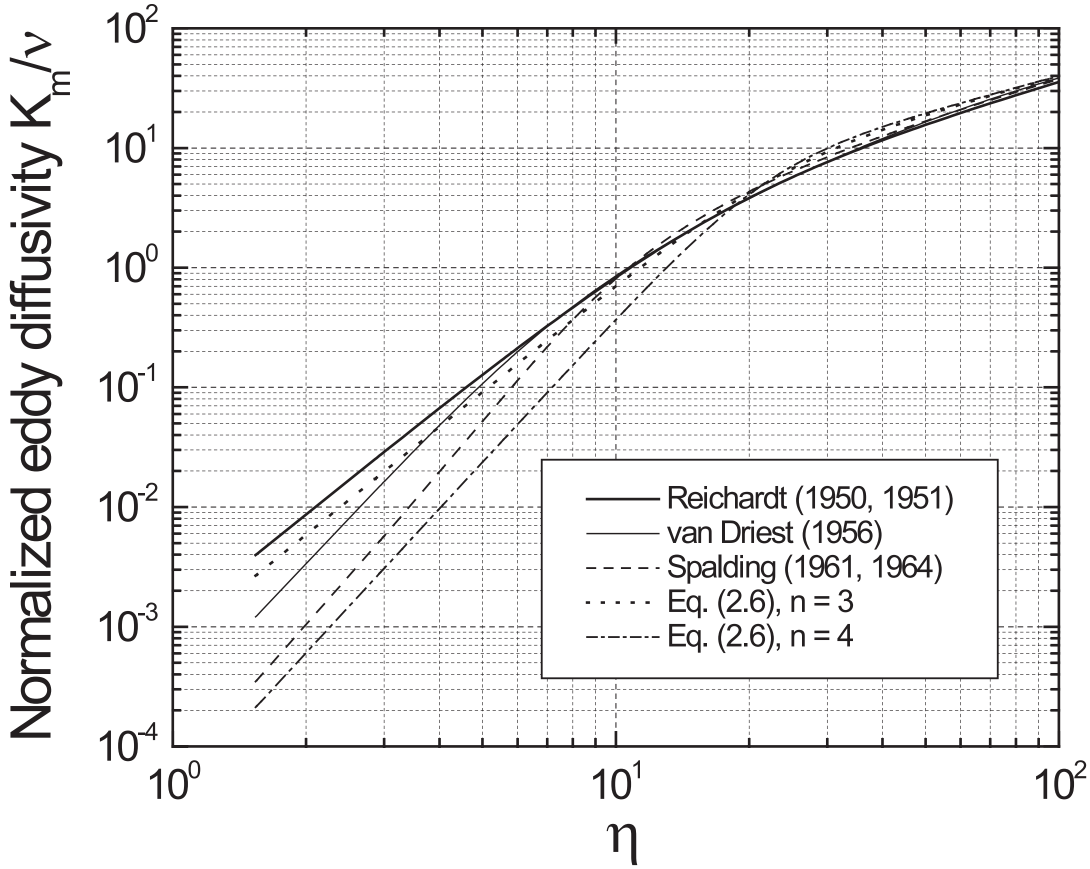


**Figure 2:** Various normalized eddy diffusivities, $K_m/\nu$, plotted against the roughness Reynolds number $\eta = u_* z/\nu$.

As shown by Kramm et al. (2002), the $K_m/\nu$-approaches of Elser (1949), Rannie (1956), and Sheppard (1958) are less satisfying because they do not fulfill Reichardt's (1951) criteria for small and, in particular, for large values of $\eta$. Consequently, these $K_m/\nu$-approaches are not discussed.

**3. Results.** Figure 1 shows the results from the numerical integration of Eq. (1.9) using the interpolation formula (2.6) with $n = 3$. As illustrated, these results fit the laboratory data of Nikuradse (1932), Reichardt (1950), and Laufer (1954) the best when $\eta_D \cong 7.35 \cdot 10^{-4}$ is used. This normalized velocity profile is called the reference velocity profile. The relative deviations of the normalized velocity profiles,

$$\delta_u = \frac{u^+|_A - u^+|_R}{u^+|_R} \times 100\% \tag{3.1}$$

provided by the various normalized eddy diffusivities, $K_m/\nu$, alternatively used (subscript $A$) and related to the reference velocity profile shown in Figure 1 (subscript $R$) are illustrated in Figure 3.

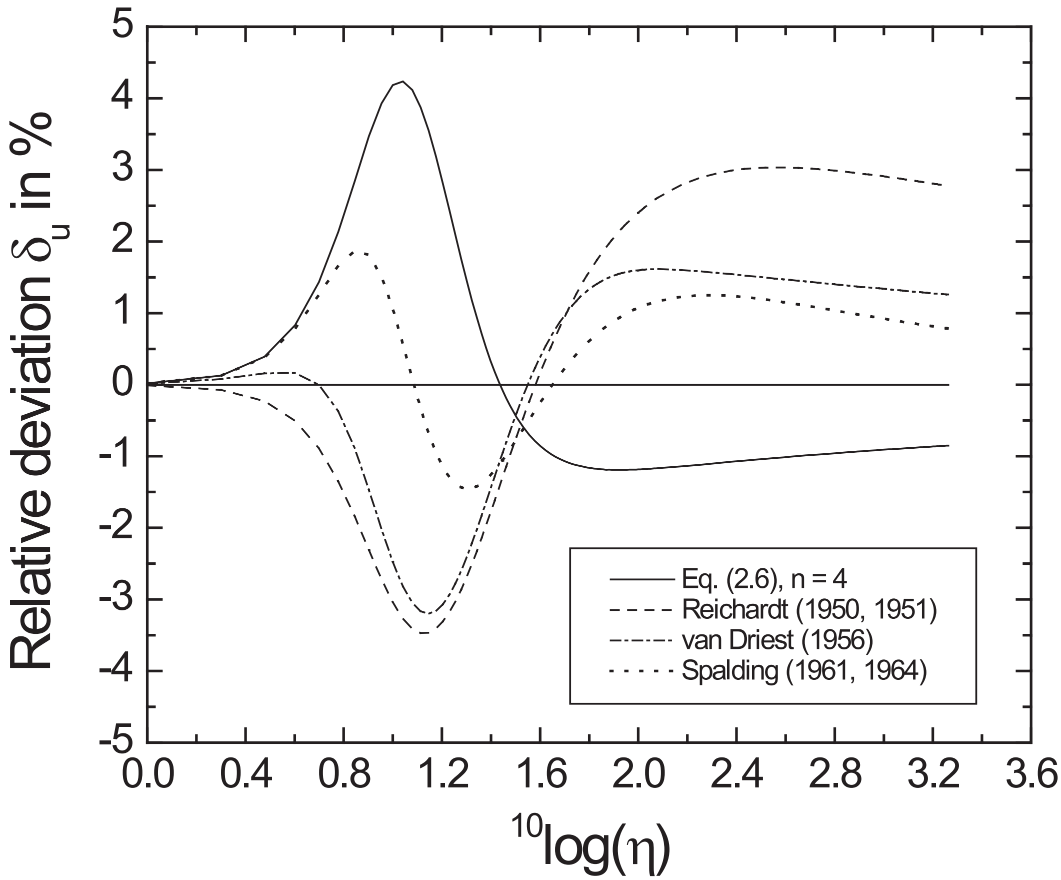


**Figure 3:** Relative deviations $\delta_u$, given by Eq. (3.1), of the normalized velocity profiles provided by the various normalized eddy diffusivities, $K_m/\nu$, alternatively used and related to the reference velocity profile illustrated in Figure 1.

Generally, all interpolation formulae elucidated here fit these laboratory data very well, but small deviations from each other occur. Compared with the reference velocity profile, the largest amounts of $\delta_u$ occur in the transition layer, followed by those in the inertial sublayer. Even though the new interpolation formula with $n = 3$, on which the reference velocity profile is based, differs from that of Spalding (1961, 1964), especially in the viscous sublayer (see Figure 2), the results obtained with both interpolation formulae only differ hardly (smaller than 2 percent), where the sign of $\delta_u$ alternates partly. In the transition layer, the relative deviations from the reference velocity profile provided by the new interpolation formula with $n = 4$ are the largest closely followed by those obtained with the $K_m/\nu$-approaches of Reichardt (1950, 1951) and van Driest (1956). These relative deviations are nearly twice as large than in the case of Spalding's (1961, 1964) interpolation formula. In the inertial sublayer, the largest amounts of $\delta_u$ are provided by Reichardt's (1950, 1951) $K_m/\nu$-approach (reflecting the influence of the constant $\eta_D$, see Eq. (2.5)) followed by that of van Driest (1956). In both cases, $\delta_u$ changes from negative values in the transition layer to positive values in the inertial sublayer. The new interpolation formula with $n = 4$ and Spalding's (1961, 1964) one provide similar small deviations, where the signs of $\delta_u$ mainly have opposite directions.

Based on these numerical results for the transfer of momentum, sublayer-Stanton numbers were derived for $Pr = 0.71$ and $Pr_t = 1$. Figure 4 shows the predicted profile of the sublayer-Stanton number for sensible heat provided by Eqs. (1.10) and (2.6) with $n = 3$ plotted against

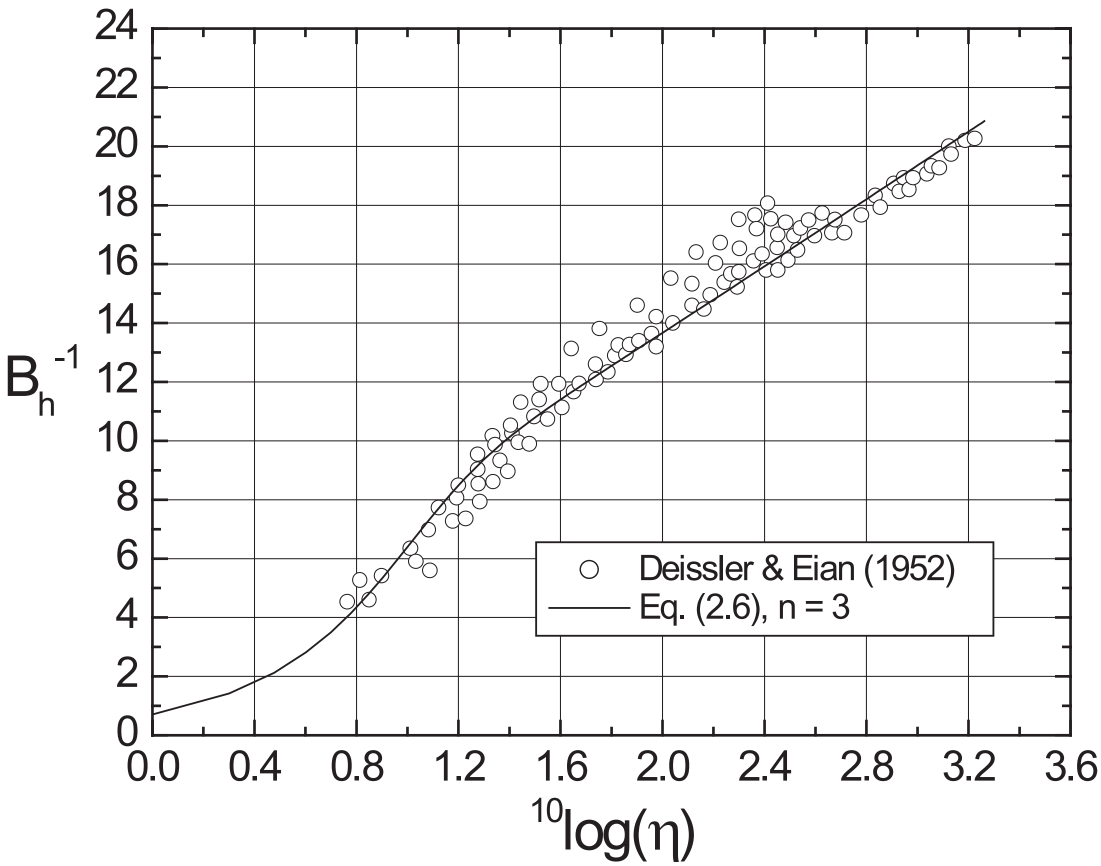


**Figure 4:** Sublayer-Stanton number profiles for the ISL over aerodynamically smooth surfaces calculated on the basis of Eq. (1.10) with $Pr = 0.71$ and $Pr_t = 1$, where the interpolation formula (2.6) is considered, and compared with the laboratory data of Deissler and Eian (1952). It is called the reference profile of the sublayer Stanton number.

the laboratory data of Deissler and Eian (1952). As illustrated in that figure, the predicted profile substantially agrees with these laboratory data. It is called the reference profile of the sublayer

Stanton number. Note that this predicted profile is not a result obtained from these laboratory data by least squares fitting. The relative deviations of the sublayer Stanton number profiles,

$$\delta_T = \frac{B_h^{-1}|_A - B_h^{-1}|_R}{B_h^{-1}|_R} \times 100\% \tag{3.2}$$

provided by the various normalized eddy diffusivities alternatively used and related to the reference profile of the sublayer Stanton number are illustrated in Figure 5. Generally, all predicted profiles of these sublayer Stanton numbers well agree these laboratory data. However, as in the case of the momentum transfer, these predicted profiles slightly differ from each other. As can be inferred from Figures 3 and 5, the amounts of the relative deviations $\delta_T$ are similar to those of $\delta_u$, where, again, the interpolation formula of Spalding (1961, 1964) provides the smallest amounts (smaller than 2 percent). As expected for the transition layer, the new interpolation formula with $n = 4$ provides

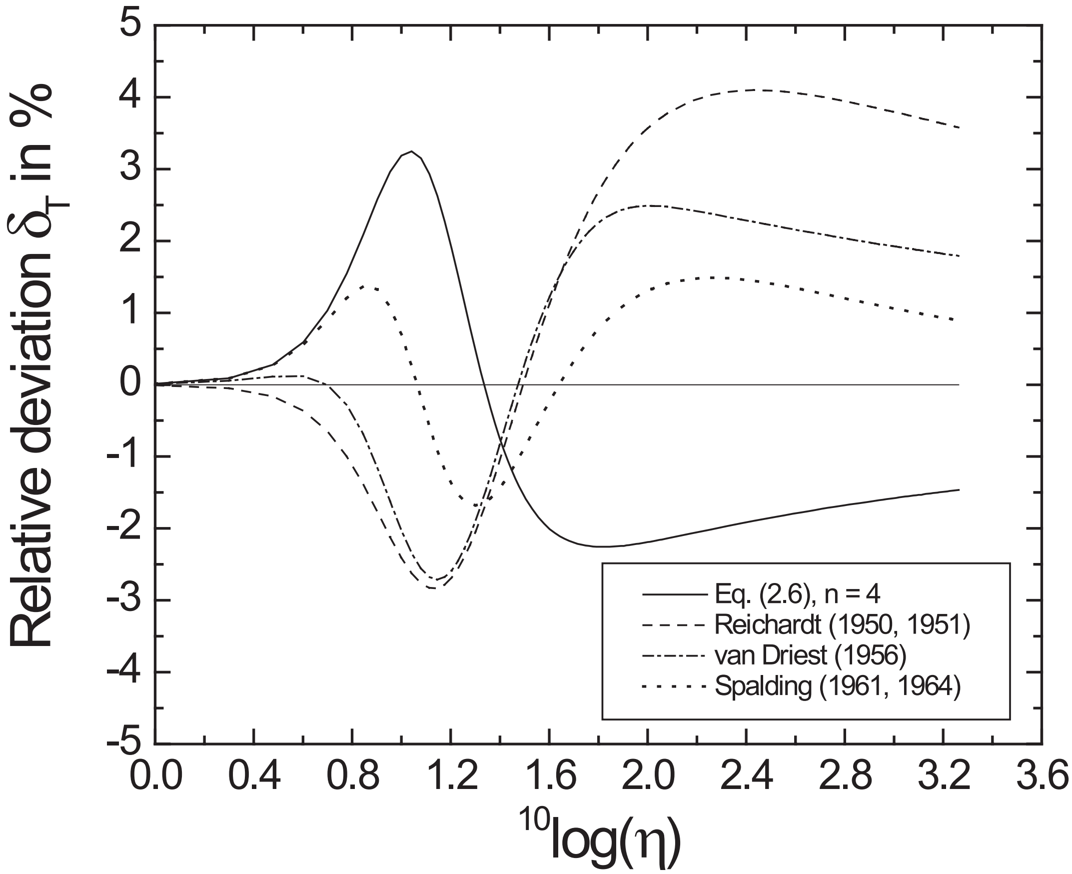


**Figure 5:** Relative deviations $\delta_T$, given by Eq. (3.2), of the sublayer Stanton number profiles provided by the various normalized eddy diffusivities, $K_m/\nu$, alternatively used and related to the reference profile of the sublayer Stanton number illustrated in Figure 4.

the largest amounts of $\delta_T$, closely followed by those that are based on the interpolation formulae of Reichardt (1950, 1951) and van Driest (1956), but all these amounts are somewhat smaller than in the case of the transfer of momentum. Again, these amounts are nearly twice as large than in the case of Spalding's (1961, 1964) interpolation formula. On the contrary, in the inertial sublayer, all amounts of $\delta_T$ become larger than in the case of the transfer of momentum, where the interpolation formula of Reichardt (1950, 1951) provides the maximum of $\delta_T$. Apparently, this maximum can be related to the influence of the constant $\eta_D$ (see Eq. (2.5)).

**4. Summary and conclusions.** The transfer of momentum, sensible heat and matter across the interfacial sublayer over aerodynamically smooth surfaces was described by using a new interpolation formula for the normalized eddy diffusivity, $K_m/\nu$, that exactly fulfills Reichardt's (1950, 1951) criteria regarding the asymptotic behavior of $K_m/\nu$ in the viscous sublayer, where it should vary with the third power of $\eta$, and in the inertial sublayer, where it should linearly depend on $\eta$, when an exponent of $n = 3$ is chosen. Results provided by numerical integration techniques using this interpolation formula were plotted against the flow velocity data of Nikuradse (1932), Reichardt (1950), and Laufer (1954) and the temperature data of Deissler and Eian (1952) for the purpose of evaluation. It was shown that the numerical results well agree with these laboratory data. As all interpolation formulae used in these study contain empirical constants, $\eta_D$, it is not surprising that in the case of the transfer of momentum the agreement between the normalized velocity profile and the laboratory data is excellent because in each case $\eta_D$ was determined from these laboratory data by least squares fitting. In contrast to this, the good agreement between the predicted profile of the sublayer Stanton number for sensible heat and the laboratory data of Deissler and Eian (1952) does not based on least squares fitting techniques applied to these laboratory data.

In addition, these numerical results were also compared with those provided by the $K_m/\nu$-approaches of Reichardt (1950, 1951), van Driest (1956), Spalding (1961, 1964), and the new interpolation formula with $n = 4$. It was shown that Reichardt's (1950, 1951) $K_m/\nu$-approach fulfills his asymptotic requirements, but not exactly. As a consequence, the corresponding relative deviations were the largest in the inertial sublayer. Even though the new interpolation formula with $n = 3$ differs from that of Spalding (1961, 1964), especially in the viscous sublayer, the results obtained with both interpolation formulae only differ hardly (smaller than 2 percent). Notably larger differences occurred in the transition layer and in the inertial sublayer when $n = 4$ was chosen or when the interpolation formulae of Reichardt (1950, 1951) and van Driest (1956) were applied.

**Appendix A: Derivation of Reichardt's criterion for the viscous sublayer.** If we consider an incompressible fluid, the equation of continuity will read

$$\nabla \cdot \mathbf{v} = 0 \tag{A1}$$

Here, $\mathbf{v}$ is the vector of the instantaneous flow velocity expressed in the sense of Reynolds by $\mathbf{v} = \bar{\mathbf{v}} + \mathbf{v}'$. Thus, we may write

$$\nabla \cdot (\bar{\mathbf{v}} + \mathbf{v}') = 0 \tag{A2}$$

Since averaging Eq. (A1) leads to $\nabla \cdot \bar{\mathbf{v}} = 0$, we can derived from Eq. (A2)

$$\nabla \cdot \mathbf{v}' = 0 \tag{A3}$$

and, hence,

$$\frac{\partial w'}{\partial z} = -\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) \tag{A4}$$

Here, $x$, $y$, and $z$ are the co-ordinates of the right-handed Cartesian frame. As mentioned in the introduction, we assume that the mean horizontal wind vector, $\overline{\mathbf{v}_H} = \bar{u}\mathbf{i}$, and the total friction stress vector, $\boldsymbol{\tau}$, are parallel, where $\mathbf{i}$ is the unit vector in $x$-direction. Under these premises we may consider the transfer of momentum as a two-dimensional problem.

To express the variation of $\bar{u}$ with $\bar{z}$ in the close vicinity of the surface we may expand it by a Maclaurin series because the velocity component vanishes at $z = 0$. In doing so, we obtain

$$\bar{u} = z\left(\frac{\partial \bar{u}}{\partial z}\right)_0 + \frac{z^2}{2}\left(\frac{\partial^2 \bar{u}}{\partial z^2}\right)_0 + \frac{z^3}{6}\left(\frac{\partial^3 \bar{u}}{\partial z^3}\right)_0 + \frac{z^4}{24}\left(\frac{\partial^4 \bar{u}}{\partial z^4}\right)_0 + \frac{z^5}{120}\left(\frac{\partial^5 \bar{u}}{\partial z^5}\right)_0 + \cdots \tag{A5}$$

At the surface, we have (see Eq. (1.1))

$$\tau_0 = \bar{\rho}\nu\left(\frac{\partial \bar{u}}{\partial z}\right)_0 \tag{A6}$$

or

$$\left(\frac{\partial \bar{u}}{\partial z}\right)_0 = \frac{\tau_0}{\nu\bar{\rho}} = \frac{u_{*,0}^2}{\nu} \tag{A7}$$

The $n$-th derivative of $\bar{u}$ can be obtained from the $(n-1)$-th differentiation of the total stress (see Eq. (1.1)) with respect to $z$ (e.g., Monin and Yaglom 1971). These derivatives are given by

$$\left(\frac{\partial u_*^2}{\partial z}\right)_0 = \nu\left(\frac{\partial^2 \bar{u}}{\partial z^2}\right)_0 - \left(\frac{\partial}{\partial z}\overline{u'w'}\right)_0 = \nu\left(\frac{\partial^2 \bar{u}}{\partial z^2}\right)_0 - \left(\overline{\frac{\partial u'}{\partial z}w'} + \overline{u'\frac{\partial w'}{\partial z}}\right)_0 \tag{A8}$$

$$\left(\frac{\partial^2 u_*^2}{\partial z^2}\right)_0 = \nu\left(\frac{\partial^3 \bar{u}}{\partial z^3}\right)_0 - \left(\frac{\partial^2}{\partial z^2}\overline{u'w'}\right)_0 = \nu\left(\frac{\partial^3 \bar{u}}{\partial z^3}\right)_0 - \left(\overline{\frac{\partial^2 u'}{\partial z^2}w'} + 2\overline{\frac{\partial u'}{\partial z}\frac{\partial w'}{\partial z}} + \overline{u'\frac{\partial^2 w'}{\partial z^2}}\right)_0 \tag{A9}$$

$$\left.\begin{aligned}\left(\frac{\partial^3 u_*^2}{\partial z^3}\right)_0 &= \nu\left(\frac{\partial^4 \bar{u}}{\partial z^4}\right)_0 - \left(\frac{\partial^3}{\partial z^3}\overline{u'w'}\right)_0 \\ &= \nu\left(\frac{\partial^4 \bar{u}}{\partial z^4}\right)_0 - \left(\overline{\frac{\partial^3 u'}{\partial z^3}w'} + 3\overline{\frac{\partial^2 u'}{\partial z^2}\frac{\partial w'}{\partial z}} + 3\overline{\frac{\partial u'}{\partial z}\frac{\partial^2 w'}{\partial z^2}} + \overline{u'\frac{\partial^3 w'}{\partial z^3}}\right)_0\end{aligned}\right\} \tag{A10}$$

and

$$\left.\begin{aligned}\left(\frac{\partial^4 u_*^2}{\partial z^4}\right)_0 &= \nu\left(\frac{\partial^5 \bar{u}}{\partial z^5}\right)_0 - \left(\frac{\partial^4}{\partial z^4}\overline{u'w'}\right)_0 \\ &= \nu\left(\frac{\partial^5 \bar{u}}{\partial z^5}\right)_0 - \left(\overline{\frac{\partial^4 u'}{\partial z^4}w'} + 4\overline{\frac{\partial^3 u'}{\partial z^3}\frac{\partial w'}{\partial z}} + 6\overline{\frac{\partial^2 u'}{\partial z^2}\frac{\partial^2 w'}{\partial z^2}} + 4\overline{\frac{\partial u'}{\partial z}\frac{\partial^3 w'}{\partial z^3}} + \overline{u'\frac{\partial^4 w'}{\partial z^4}}\right)_0\end{aligned}\right\} \tag{A11}$$

and so on. If the friction velocity is invariant with height, i.e., $u_* = u_{*,0}$, the left sides of Eqs. (A8) to (A11) will be equal to zero. In such a case we have (e.g., Monin and Yaglom 1971)

$$\left(\frac{\partial^2 \bar{u}}{\partial z^2}\right)_0 = \frac{1}{\nu}\left(\overline{\frac{\partial u'}{\partial z}w'} + \overline{u'\frac{\partial w'}{\partial z}}\right)_0 \tag{A12}$$

$$\left(\frac{\partial^3 \bar{u}}{\partial z^3}\right)_0 = \frac{1}{\nu}\left(\overline{\frac{\partial^2 u'}{\partial z^2}w'} + 2\overline{\frac{\partial u'}{\partial z}\frac{\partial w'}{\partial z}} + \overline{u'\frac{\partial^2 w'}{\partial z^2}}\right)_0 \tag{A13}$$

$$\left(\frac{\partial^4 \bar{u}}{\partial z^4}\right)_0 = \frac{1}{\nu}\left(\overline{\frac{\partial^3 u'}{\partial z^3}w'} + 3\overline{\frac{\partial^2 u'}{\partial z^2}\frac{\partial w'}{\partial z}} + 3\overline{\frac{\partial u'}{\partial z}\frac{\partial^2 w'}{\partial z^2}} + \overline{u'\frac{\partial^3 w'}{\partial z^3}}\right)_0 \tag{A14}$$

and

$$\left(\frac{\partial^5 \bar{u}}{\partial z^5}\right)_0 = \frac{1}{\nu}\left(\overline{\frac{\partial^4 u'}{\partial z^4}w'} + 4\overline{\frac{\partial^3 u'}{\partial z^3}\frac{\partial w'}{\partial z}} + 6\overline{\frac{\partial^2 u'}{\partial z^2}\frac{\partial^2 w'}{\partial z^2}} + 4\overline{\frac{\partial u'}{\partial z}\frac{\partial^3 w'}{\partial z^3}} + \overline{u'\frac{\partial^4 w'}{\partial z^4}}\right)_0 \tag{A15}$$

Naturally, $u' = v' = w' = 0$ at the surface ($z = 0$). If we assume that all derivative of the velocity fluctuations with respect to $x$ and $y$ are equal to zero, we can infer from Eq. (A4) that $(\partial w'/\partial z)_0 = 0$ (e.g., Monin and Yaglom 1971). Under these premises we obtain

$$\left(\frac{\partial^2 \bar{u}}{\partial z^2}\right)_0 = 0 \tag{A16}$$

$$\left(\frac{\partial^3 \bar{u}}{\partial z^3}\right)_0 = 0 \tag{A17}$$

$$\left(\frac{\partial^4 \bar{u}}{\partial z^4}\right)_0 = \frac{3}{\nu}\left(\overline{\frac{\partial u'}{\partial z}\frac{\partial^2 w'}{\partial z^2}}\right)_0 \tag{A18}$$

and

$$\left(\frac{\partial^5 \bar{u}}{\partial z^5}\right)_0 = \frac{1}{\nu}\left(6\overline{\frac{\partial^2 u'}{\partial z^2}\frac{\partial^2 w'}{\partial z^2}} + 4\overline{\frac{\partial u'}{\partial z}\frac{\partial^3 w'}{\partial z^3}}\right)_0 \tag{A19}$$

Introducing Eq. (A7) and the expressions (A16) to (A19) into Eq. (A5) yields then

$$\bar{u} = \frac{u_*^2}{\nu}z + \frac{1}{8\nu}\left(\overline{\frac{\partial u'}{\partial z}\frac{\partial^2 w'}{\partial z^2}}\right)_0 z^4 + \frac{1}{120\nu}\left(6\overline{\frac{\partial^2 u'}{\partial z^2}\frac{\partial^2 w'}{\partial z^2}} + 4\overline{\frac{\partial u'}{\partial z}\frac{\partial^3 w'}{\partial z^3}}\right)_0 z^5 + \cdots \tag{A20}$$

or

$$\frac{\bar{u}}{u_*} = \eta + \frac{\nu^3}{8u_*^5}\left(\overline{\frac{\partial u'}{\partial z}\frac{\partial^2 w'}{\partial z^2}}\right)_0 \eta^4 + \frac{\nu^4}{120u_*^6}\left(6\overline{\frac{\partial^2 u'}{\partial z^2}\frac{\partial^2 w'}{\partial z^2}} + 4\overline{\frac{\partial u'}{\partial z}\frac{\partial^3 w'}{\partial z^3}}\right)_0 \eta^5 + \cdots \tag{A21}$$

This equation complete agrees with that of Monin and Yaglom (1971). From Eq. (1.3) we obtain (see also Reichardt 1951)

$$\frac{\partial}{\partial \eta}\left(\frac{\bar{u}}{u_*}\right) = \left(1 + \frac{K_m}{\nu}\right)^{-1} \tag{A22}$$

In the viscous sublayer the value of the normalized eddy diffusivity is much smaller than unity (see Figure 2). Therefore, Eq. (A22) may be written as

$$\frac{\partial}{\partial \eta}\left(\frac{\bar{u}}{u_*}\right) \cong 1 - \frac{K_m}{\nu} \tag{A23}$$

On the other hand, the derivative of Eq. (A21) with respect to $\eta$ is given by

$$\frac{\partial}{\partial \eta}\left(\frac{\bar{u}}{u_*}\right) = 1 + \frac{\nu^3}{2u_*^5}\left(\overline{\frac{\partial u'}{\partial z}\frac{\partial^2 w'}{\partial z^2}}\right)_0 \eta^3 + \frac{\nu^4}{24u_*^6}\left(6\overline{\frac{\partial^2 u'}{\partial z^2}\frac{\partial^2 w'}{\partial z^2}} + 4\overline{\frac{\partial u'}{\partial z}\frac{\partial^3 w'}{\partial z^3}}\right)_0 \eta^4 + \cdots \tag{A24}$$

Combining Eqs. (A23) and (A24) yields finally

$$\frac{K_m}{\nu} = -\frac{\nu^3}{2u_*^5}\left(\overline{\frac{\partial u'}{\partial z}\frac{\partial^2 w'}{\partial z^2}}\right)_0 \eta^3 - \frac{\nu^4}{24u_*^6}\left(6\overline{\frac{\partial^2 u'}{\partial z^2}\frac{\partial^2 w'}{\partial z^2}} + 4\overline{\frac{\partial u'}{\partial z}\frac{\partial^3 w'}{\partial z^3}}\right)_0 \eta^4 - \cdots \geq 0 \tag{A25}$$

By considering the entropy balance for turbulent atmospheric flows, Herbert (1975) showed that such eddy diffusivities must always be positive definite. The right-hand side of Eq. (A25) must, therefore, be positive definite. Note that Monin and Yaglom (1971) intuitively argued that close to the wall the first term,

$$\left(\overline{\frac{\partial u'}{\partial z}\frac{\partial^2 w'}{\partial z^2}}\right)_0 = \left(\frac{\partial^3}{\partial z^3}\overline{u'w'}\right)_0 < 0 \tag{A26}$$

should be smaller than zero.

Even thought the derivation of the Eq. (A25) strongly differs from those presented by Reichardt (1951) and Elrod (1957), this equation reflects Reichardt's criterion for the viscous sublayer in an excellent manner.

As indicated, the normalized eddy diffusivity should, at least, vary with the third power of $\eta$ when $\eta$ approaches to zero. An $\eta^4$-behavior as reflected by the interpolation formulae of van Driest (1956) and Spalding (1961, 1964) (and Deissler 1955) and the new one with $n = 4$ would prevail when $\partial u'/\partial z$ and $\partial^2 w'/\partial z^2$ are uncorrelated.

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