Gravitational and Inertial Mass

Outline
1. Einstein’s Principle of Equivalence
2. Newton’s experiment
3. Eötvös-Dicke experiments
4. Implications for General Relativity

Einstein’s Principle of Equivalence

Einstein, while thinking about bodies falling in a gravitational field, realized that to a falling observer there is no gravitational field. That is, the observer can do a local experiment such as dropping an object and observe that the object does not move in his frame. This observer might interpret this experiment to show that he is at rest.

Following Berry\(^1\), consider a region where only a gravitational field \(\vec{g}\) exists. In a stationary frame all masses would fall at the same rate: \(m\vec{a} = m\vec{g}\). Now if we transform to a frame \(f'\) that is accelerating at \(\vec{g}\) a body will appear stationary unless a non-gravitational force is present. In this case \(m\vec{a} = m\vec{g} + \vec{F}_{ng}\), where \(F_{ng}\) represents the non-gravitational forces present. Since \(\vec{a}' = \vec{a} - \vec{g}\) we have \(ma' = \vec{F}_{ng}\). In the frame \(f'\) gravitational forces do not appear; the frame is in ‘free-fall’.

Thus, the equality of inertial and gravitational mass implies that in a laboratory falling in a gravitational field the laws of physics are the same as those observed in a Newtonian inertial frame with no gravity. This is Einstein’s principle of equivalence. This insight lead Einstein to develop General Relativity.

Newton’s experiment

Galileo’s experiments with falling bodies had already indicated the equivalence of gravitational and inertial masses. Newton understood this and took it upon himself to investigate the relationship more carefully. He constructed two pendulums eleven feet long with small cylindrical containers forming the bob of each pendulum\(^2\). He placed equal weights of various substances, including gold, silver, lead, glass, sand, wood, water and wheat, into the containers and set the pendulums in oscillation. He states: “In these experiments, in bodies of the same weight, a difference of matter that would be even less than a thousandth part of the whole could have been clearly noticed”.

His reasoning was as follows\(^3\). Using the second law of motion, the change in velocity of the inertial mass in a time \(\Delta t\) results from the action of the gravitational force and geometrical factors: \(\Delta v = F(\theta)\Delta t/m_1\). The motive force, \(F(\theta)\), is proportional to the weight of the bob. Thus, the ratio of the motive forces operating on two, identical pendulums is the ratio of the weights: \(F_1(\theta)/F_2(\theta) = W_1/W_2\). Using this, the second law can be written to give the ratio of the velocity changes as

\[
\frac{\Delta v_1}{\Delta v_2} = \frac{W_1m_{12}\Delta t_1}{W_2m_{11}\Delta t_2},
\]  

(1)

Further, since \(\Delta v = L\Delta \theta/\Delta t\) the same changes in \(\theta\) occur for both pendulums, or

\[
\frac{\Delta \theta_1}{\Delta \theta_2} = \frac{W_1m_{12}}{W_2m_{11}} \left(\frac{\Delta t_1}{\Delta t_2}\right)^2.
\]  

(2)

\(^{1}\) Berry, M. V., Principles of Cosmology and Gravitation, IOP Publishing, 1993

\(^{2}\) Principia, Book III, Proposition VI, Theorem VI, F. Cajori, Univ. of California Press, 1973

\(^{3}\) http://www.mathpages.com
When the experiment is performed, the time taken for a large number of complete oscillations of the pendulums allows us to reduce equation (2) to the form

$$\frac{m_1}{m_2} = \frac{W_1}{W_2} \left( \frac{\Delta t_1}{\Delta t_2} \right)^2.$$  \hspace{1cm} (3)

Newton found the equivalence of gravitational and inertial masses to approximately one part in a thousand. The French mathematician, Bessel, improved on the experimental arrangement using pendulums and increased the accuracy to $\sim 1/60,000$.

### The Eotvos Dicke Experiments

Baron Roland von Eötvös began a series of measurements in 1906 that was to improve the accuracy of the equivalence of inertial and gravitational masses to $\sim 1/20,000,000$. His method was based upon a torsional pendulum in which two masses were suspended at differing distances from a wire. In this experiment each mass is subject to the gravitational force of the earth and to the acceleration it undergoes as the earth rotates. The experiment was repeated and refined by a number of investigators, most recently by R. Dicke who improved the apparatus to account for some of the uncontrolled error sources in Eötvös' original experiment.

The figure at the right shows a sketch of the forces on a mass in the Eötvös apparatus. The gravitational force $W = m_g g$ acts vertically downward while the centripetal acceleration requires a force $F_c = m_c a_c$ where $a_c = R \Omega^2$, $R$ is the Earth's radius and $\Omega$ is the Earth's angular speed. At equilibrium, the vertical forces balance:

$$L_1(m_1 a_c \cos \lambda - m_1 g) = L_2(m_2 a_c \cos \lambda - m_2 g) \hspace{1cm} (4)$$

where $L_1$ and $L_2$ are the distances of the two masses from the wire suspension.

The horizontal forces produce a torque on the wire of

$$\tau = L_1 m_1 a_c \sin \lambda - L_2 m_2 a_c \sin \lambda. \hspace{1cm} (5)$$

Combining equations (4) and (5) leads to the expression

$$\tau = L_1 m_1 G(\lambda) \left[ 1 - \left( \frac{F(\lambda) - \frac{m_1 \lambda}{m_1 g}}{F(\lambda) - \frac{m_2 \lambda}{m_2 g}} \right) \right] \hspace{1cm} (6)$$

where $F(\lambda) = a_c \cos \lambda$ and $G(\lambda) = a_c \sin \lambda$. Equation (6) can be simplified by noticing that $F(\lambda) \ll g$. With this assumption the horizontal torque becomes:

$$\tau = C \frac{m_2}{m_1} \frac{m_2}{m_1} \hspace{1cm} (7)$$

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Equation (7) shows that if the inertial and gravitational masses are not identical that there will be a net horizontal torque on the pendulum and the masses will begin to rotate. The actual experimental procedure involved exchanging the masses periodically to force an oscillatory motion of the pendulum. Eötvös repeated the experiment with different materials over several years.

Dicke commented on possible errors in the work by Eötvös and subsequent workers and then performed his own experiment. By 1964 Dicke and his co-workers reported the equivalence of the inertial and gravitational masses to 1 part in $10^{-11}$. There have been other, more recent, estimates giving a similar results\(^5\).

In the Newtonian theory of gravity this appears to be a remarkable coincidence. However, in General Relativity it is taken as a fundamental property of the curvature of spacetime.

*Homework: Derive equations (3) and (7)*

\(^5\) See, for example, the discussion in R. Hakim, An Introduction to Relativistic Gravitation, Cambridge University Press, 1999