Sea Ice Deformation and Rheology

Jenny Hutchings

X7569

jenny@iarc.uaf.edu
Recap Force Balance

\[ \frac{\partial mU}{\partial t} + \nabla \cdot (mUU) = \tau_a + \tau_w - mf \hat{k} \times U - mg \nabla H + \nabla \cdot \sigma, \]

Fig. 8. Mean balance of forces over a 60-hour interval.
Deformation

Deformations results from stresses within the continuum induced by external forces or due to changes in its temperature. The relation between stresses and induced strains is expressed by constitutive equations, e.g. Hooke's law for linear elastic materials.
Strain

- Deformation is the change in shape and/or size due to applied force.
- It is described by strain, or strain rate.

Engineering (Cauchy) strain is the change in length per unit length.

\[ \dot{\varepsilon} = \frac{1}{2} \left[ \nabla \mathbf{U} + \nabla \mathbf{U}^T \right] \]

\[
\begin{bmatrix}
\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\
\varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\
\varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial u_x}{\partial x} & \frac{1}{2} \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) & \frac{1}{2} \left( \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) \\
\frac{1}{2} \left( \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \right) & \frac{\partial u_y}{\partial y} & \frac{1}{2} \left( \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) \\
\frac{1}{2} \left( \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} \right) & \frac{1}{2} \left( \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} \right) & \frac{\partial u_z}{\partial z}
\end{bmatrix}
\]
Strain and Kinematics

\[ \frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y}, \quad \frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y}, \quad \frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial y}, \quad \frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y} \]

Divergence

Vorticity

Pure Shear

Normal Shear

Strain Invariants

\[ \varepsilon_I = \frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y}, \quad \varepsilon_{II} = \frac{1}{2} \sqrt{ (\frac{\partial U_x}{\partial x} - \frac{\partial U_y}{\partial y})^2 + (\frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x})^2 } . \]
Estimating Strain rate

- Given a set of points on the ice pack, that can be tracked in time and surround a region of ice, the strain-rate can be estimated numerically.

\[ u = \frac{dx}{dt} \sim \frac{(x_1-x_2)}{T} \]

\[ Du/dx \sim \frac{(u_a-u_b)}{X} \]
RGPS Observations

- Divergence, vorticity and shear from RADARSat ScanSAR analysis.

Kwok
Fall, Winter, Spring 97-98
Typical Failure Modes (Laboratory stress tests)

Figure 2. Typical failure modes; in compression, (a) longitudinal splitting, (b) shear fracture, and (c) multiple shear fractures and, in tension, (d) extension fracture.
Internal Ice Stress

\[ \frac{\partial mU}{\partial t} + \nabla \cdot (mUU) = \tau_a + \tau_w - mf\hat{k} \times U - mg\nabla H + \nabla \cdot \sigma. \]

\[ F_i = \frac{\partial \sigma_{ij}}{\partial \xi_j} \]

• We need to relate internal ice stress, \( \sigma \), to velocity, \( U \), or strain rate, \( \varepsilon \), in order to solve the momentum balance.
• Typically an empirical model is used, that describes the constitutive law relating stress to strain rate.
• Often referred to as rheology.
Stress

- $\sigma$ is a stress tensor, which describes the stress at a point in three orthogonal directions.
- extent of sea ice $>>$ thickness, so we reduce stress tensor to 2 dimensions, x and y.
- Components of stress are $\sigma_{xx}$, $\sigma_{yy}$, $\sigma_{xy}$, $\sigma_{yx}$

Stress Invariants:

- $\sigma_I$ compressive
- $\sigma_{II}$ shear

Principle Stresses:

- $\sigma_1 = \sigma_I + \sigma_{II}$
- $\sigma_2 = \sigma_I - \sigma_{II}$

Simple 1-D case:
Stress has 1 component (scalar)
Stress Invariants

- $\sigma_1 = \sigma_{xx} + \sigma_{yy}$
- $\sigma_{\text{II}} = \frac{1}{2} \left[ (\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{xy} + \sigma_{yx})^2 \right]^{1/2}$
- Often assumed $\sigma_{xy} = \sigma_{yx}$
- $\sigma_{\text{II}} = \frac{1}{2} \left[ (\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2 \right]^{1/2}$
Classic Rheological Models

- Linking stress ($\sigma = F/A$) to the strain $\varepsilon = (l-l_0)/l_0$

- Elastic (solid): $\sigma = E\varepsilon$ with Young’s modulus $E$

- Newtonian liquid: $\sigma = \eta d\varepsilon/dt$

- Plastic (solid): $\sigma = k$, for $\varepsilon > 0$ $\varepsilon = 0$, for $\sigma < k$
How does pack ice break?

- Sea ice is a compressible solid that deforms through brittle failure.
- On geophysical scales O(1-1000km) there is no evidence failure is ductile or viscous.
Elastic - Plastic
AIDJEX Assumptions

1. Enough leads are present in a 100 km by 100 km region to make the ice isotropic on that scale.

2. The ice behavior could be approximated by an isotropic yield surface. I.e. material strength is not dependent on direction, such that stress state can be projected onto a 2-D principle space described by stress invariants.

3. The ice had no tensile strength. I.e. it is highly fractured and has little resistance to divergence.

4. Compressive ice strength independent of strain rate. I.e. There is a constant ice strength under compression.
Constitutive Relations

• Low tensile strength.
• Ice supports shear stresses.
• Ice strength can be related to ice thickness and area, but is independent of strain rate.

Elastic-Plastic, Pritchard 1976
Granular Plastic, Coon 1979
Coulombic Rheology

Figure B2. Mohr's circle constructions for (a) the stress and (b) the strain rate of the thin ice in the finite width flaw shown in Figure 3b.

Figure B1. Thin ice coulombic yield curve used for analytical calculations.

\[ |\tau_m| = a|\sigma_m| + 1 \]

\[ a = \cos(2\theta'_c) \]
Viscous-Plastic Model

- Hibler 1979

Plastic

\[
\sigma = 2\eta \dot{\varepsilon} + I \left[ (\zeta - \eta) \text{tr} (\dot{\varepsilon}) - \frac{P}{2} \right],
\]

\[
\zeta = \frac{P}{2\Delta}, \quad \eta = \frac{\zeta}{c^2},
\]

\[
\Delta = \left[ \text{tr} (\dot{\varepsilon})^2 + \frac{2}{c^2} \dot{\varepsilon} : \dot{\varepsilon} \right]^{\frac{1}{2}}.
\]

Viscous

\[
\zeta_{\text{max}} = 2.5 \times 10^8 P, \quad \eta_{\text{max}} = \frac{\zeta_{\text{max}}}{c^2}.
\]

Figure 2.1: The elliptical yield curve, with \( P = 2 \), normalised by \( P \)

\[
\sigma_I = \zeta \dot{\varepsilon}_I - \frac{P}{2}
\]

\[
\sigma_{II} = \eta \dot{\varepsilon}_{II}
\]
Assorted Yield Criteria
Shear Stress and Ice Arches
Ip’s results

\[ 0 = \rho_a c_a u_g^2 - \rho_a c_w u_w^2 + \left( \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} \right) \]

\[ 1 = \beta - \gamma \left( \frac{\partial \sigma_{x'x'}}{\partial x'} + \frac{\partial \sigma_{x'y'}}{\partial y'} \right) \]

\[ \beta = \frac{\rho_w c_w u_w^2}{\rho_a c_a u_g^2} \quad \gamma = \frac{P}{\lambda \rho_a c_a u_g^2} \]
Ip’s results for different yield surfaces
Ice Strength

- For isotropic ice this is the compressive stress at failure.

Fig. 3.2.2.6: Stress/strain-rate plot for granular sea ice (Sanderson, 1988).
Estimating ice strength

- in a simplified approach, assuming that the ice momentum balance is controlled solely by wind and current forcing, the forces that build up within the ice cover are dependent on the wind (\(a\)) and current (\(w\)) shear stress \(\tau_{w,a}\), which is given by (see Section 3.1 for more details)

\[
\tau_{w,a} = c_D \rho_{w,a} |u|^2
\]

- taking \(c_D\) as \(1.3 \times 10^{-3}\) for both the upper and lower surface and the appropriate densities for water and air, the magnitude of the shear stress is shown in Fig. 3.2.3.1

- the in-situ magnitude of the shear stress depends on the length of fetch \(L\) over which the stress may build up, such that the local force per unit width of ice \(F'\) is

\[
F' = L \tau
\]
- based on Fig. 3.2.3.1, for an ice cover of 1 m thickness this results in values which are far below typical macroscopic ice strengths and would not result in failure even if operating over a long period of time, i.e. after accumulation of considerable strain; Parmerter and Coon (1972), however, found values for $F_1$ around 1 kN m$^2$ during ridge building, with $\tau$ around 0.05 N m$^2$ and $L$ around 20 km.
Ice Strength: Estimate on geophysical scale

Steady wind, no ice motion.
Lower bound on strength of $1 \times 10^5 \text{ Nm}^{-1}$
Strength

- Strength varies according to the scale it is measured over.
- Laboratory samples: $\sim 10\text{MPa}$ ($10^7 \text{ Pa}$)
- Geophysical scale, lower bound estimate is $\sim 10-100 \text{kPa}$ ($10^4 - 10^5 \text{ Pa}$)

Figure 1.4: (a) Geometrically similar structures of different sizes $D$ and (b) power scaling laws

Bazant
Laboratory Estimates of Brittle Failure Envelope

Figure 4a. Brittle failure envelope for first-year S2 sea ice loaded across the columns (i.e., along directions $X_1$ and $X_2$) at $-10^\circ$C. The shape of the points denotes the mode of failure.

Schulson et al. 2006
Tensile Strength

• Hibler + Schulson Modified Coulombic yield surface.

Figure A1. The laboratory coulombic yield curve for thin and thick ice for $P^* = 10^3 \, \text{N m}^{-1}$. 
Figure 3. Coulomb failure surface determined from upper bound of measured ice stress. NWRA is North-West Research Associates, Inc.; SIMI is Sea Ice Mechanics Initiative.
Can Laboratory results be applied to the Geophysical Scale?

Indications that brittle ice failure is scale invariant from m – 1000km scales:

• Apparent scale invariance of deformation observed on scales of 10km-1000km (Marsan & Weiss)

• Apparent similarity between cracks in laboratory samples and ice pack (Schulson)
Cracks
Wing Cracks
Leads in Ice Pack
VP Model

Figure 6. Contrast between the net winter deformation (divergence, vorticity, and shear) from model simulations and RGPS ice drift. (a) NPS, (b) PIOMAS, (c) ECCO2, (d) NPS, and (e) LANL (strain rate units: d^{-1}).
Fractile properties of strain rate

FIG. 1 (color online). Sea-ice deformation rate on 6 November 1997 from 42571 RGPS cells.

FIG. 2. Total deformation rate $\dot{\epsilon}_L$ as a function of scale $L$ (81586 samples). Vertical dashes define bins. Gray dots are means within each bin. Gray solid line is least squares fit to mean values. $A$ is 13–20 km scale; $B$ is 160–320 km scale.
• VP model does not simulate largest strain rate events.

Girard et al. 2009, following Marsan et al. 2004

FIG. 3. Cumulative probability of observing a given deformation rate, for the 13–20 km scale (A) and the 160–320 km scale (B). Slope of dashed lines: −2.5 (A) and −3.6 (B). The slope becomes shallower as the scale decreases.
Why should we care about this?

- Leads grow and ridge ice. Are they a significant player in the ice mass balance?
- Ocean-atmosphere fluxes are concentrated at leads in winter. Models should correctly reproduce lead area.
- Shearing ridges might be a significant player in kinetic energy transfer to ocean.
Kinetic Energy to Ocean

- Ice interaction damps energy transfer to ocean.
- However large shear events cause upwelling.

McPhee et al. 2005
Fig. 26. Schematic sketch of two types of deformation: type-I depicts ice as a continuum; type-II, as a granular material. A fault/lead develops during terminal failure (b,c), creating gouge (c) across which frictional sliding occurs at lower applied stresses. Eventually, the fault/lead heals or the wind changes direction (d) and the ice again behaves as a continuum.
Bill Hibler considers each grid cell is composed of lead ice and pack ice. He calculates stress state as a composite value of lead and pack ice, and with two equations he can find the two unknowns – lead width and orientation.

Figure 1. Schematic of oriented thin ice lead or flaw embedded in thick ice. The isotropic yield curve for both the thin and thick ice is shown in the insert.
Max Coon points out that the oriented ice within a lead should be described by 3 stress values (normal stresses along and perpendicular to lead, and shear stress along the lead).
Practicalities

• Numerical modelling of a 3-D yield surface is overly expensive.
• Granular or particle models might “be the buisness”!
• In these models a cascade of weakening allows large fractures to emerge.
• Constitutive relationship is provided to govern particle interaction.
Elastic-Brittle Model
Ice Strength – Models

\[ P = P^* h \exp \left[ -C (1 - A) \right], \]

Allowing failure stress to increase as ice thickens, approximates strain hardening.

As \( P \) is exponentially related to \( A \), force balance approaches free drift quickly as \( A \) decreases.