Modification of emission from the X-mode at the third electron cyclotron harmonic from mode conversion and reflection

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Both absorption and emission of the X-mode in the vicinity of the third electron cyclotron harmonic ($\omega \sim 3\omega_{ce}$) in a weakly inhomogeneous magnetic field are analyzed. The mode conversion problem is solved both without absorption and with synchrotron absorption and emission. The transmission is shown to be independent of absorption, and the emission is related to absorption exclusively through the reflection coefficient, which is calculated. The numerical values for the reflection coefficient are obtained from a sum that is exact, and quickly gives high accuracy values within its radius of convergence. The effects of reflection are shown to be generally neglectible for most laboratory plasmas, but potentially important in some space plasmas. © *1995 American Institute of Physics.*

I. INTRODUCTION

Since it has been determined that electron cyclotron emission is affected by the reflection coefficient for emission on the low magnetic field side of a cyclotron harmonic layer,[1] it has been of interest to obtain quantitative estimates of the potential error in neglecting these effects. At the second electron cyclotron harmonic, estimates have been obtained,[2] and the error estimates are generally small, but for laboratory plasmas, part of the reason is that most electron cyclotron emission (ECE) diagnostics operate at $2\omega_{ce}$, where the plasma is very nearly a blackbody emitter. In these cases, the subtleties of reflection and transmission are usually negligible. For ECE emission near $3\omega_{ce}$, however, even fusion plasmas are not completely blackbody emitters, so as the employment of this diagnostic becomes more common, the question again arises over the potential error due to the typical neglect of the reflection coefficient.

Even at the second harmonic, the reflection coefficient for synchrotron radiation was nontrivial to calculate until the advent of the new technique that obtains the reflection coefficient to unprecedented accuracy and speed from the summation of a series rather than the solution of a tunneling equation.[2] Similar kinds of calculations have been done for both the second and third harmonic for a five-branch case with finite $k_\parallel$ (the O-mode and X-mode are coupled via the mode converted Bernstein wave in a five-branch problem), first using an integral equation approach,[3,4] and more recently the series approach.[5] This latter result gave an approximate formula that is indicative, but not accurate enough to make error estimates.

In this paper, the complications of the five-branch problem are avoided by letting $k_\parallel = 0$, so that only the X-mode with weak relativistic absorption is considered. Although the analysis of the $3\omega_{ce}$ mode conversion problem is similar to an earlier analysis of the $3\omega_{ci}$ mode conversion problem,[6] there are enough differences that one must start from the beginning. The first step is to formulate the appropriate tunneling equation from the dispersion relation, then solve it without absorption to find the basic coupling between the propagating branches. Using the Green's function formalism, the original differential equation is then cast into the form of an integral equation whose asymptotic solutions give the scattering parameters in terms of integrals. Although the integrands involve the solutions of the original differential equation or of the integral equation, their asymptotic forms may be represented by a series that may be exactly integrated term by term, and the result for the reflection coefficient is merely the sum of these integrals whose terms take the form of a power series in the absorption parameter. The results of the numerical evaluation of the sum of the series is then summarized in terms of semi-empirical formulas, which accurately represent the results in terms of three dimensionless parameters over a broad range for each parameter. The conclusion summarizes some general features observed about the reflection coefficient and its importance, and gives an upper bound on the potential error in ECE diagnostics if one ignores the reflection. It is also noted that other estimates of the reflection coefficient that are not based on the mode conversion analysis are not accurate enough for practical use.

II. THE DISPERSION RELATION AND TUNNELING EQUATION

For the X-mode where $\omega \sim 3\omega_{ce}$, the relevant dielectric tensor elements for a weakly relativistic plasma may be written as[7]

$$K_1 \simeq 1 + \frac{\omega_{pe}^2}{\omega_{ce}^2 - \omega^2} - \frac{9\omega_{pe}^2}{\omega^2}\frac{\mu I_3(\lambda)}{\lambda} F_{9/2}[\mu(1-3\omega_{ce}/\omega)]$$

$$\simeq 1 - \frac{9X}{8} + \frac{3XL}{4(x-x_0)}\left(\frac{9D^2}{2\mu}\right)^2 F, \tag{1}$$

$$K_2 \simeq i\frac{\omega_{pe}^2\omega_{ce}}{\omega(\omega^2-\omega_{ce}^2)} + \frac{9i\omega_{pe}^2}{\omega^2}\frac{\mu I_3(\lambda)}{\lambda} F_{9/2}[\mu(1-3\omega_{ce}/\omega)]$$

$$\simeq \frac{3iX}{8} - \frac{3iXL}{4(x-x_0)}\left(\frac{9D^2}{2\mu}\right)^2 F, \tag{2}$$

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where $X=\omega_{pe}^2/\omega^2$, $\mu=m_e c^2/\mathscr{K}T_e$, $\lambda=\frac{1}{2}k_\perp^2\rho_{Le}^2$, $\mu(1-3\omega_{ce}/\omega)=-\mu x/L$, since we assume $B=B_0(1+x/L)$, so that L is the scale length for the variation of the magnetic field, $F\equiv\zeta F_{9/2}(\zeta-\frac{9}{2})$, where F_q is the weakly relativistic plasma dispersion function,[8] $D^2\equiv -k_\perp^2c^2/\omega^2$, and $x_0=9L/2\mu$. The dispersion relation for the X-mode, neglecting terms of order λ that are nonresonant, is then given by

$$\frac{k_\perp^2c^2}{\omega^2}=\frac{K_1^2+K_2^2}{K_1}. \tag{3}$$

Using the expressions for the dielectric tensor elements from Eqs. (1) and (2), the dispersion relation may be written as a cubic in D^2, so that

$$D^6+2\left(1-\frac{3X}{4}\right)D^4+\beta\left(1-\frac{9X}{8}\right)(x-x_0)D^2$$
$$+\beta\left(1-\frac{3X}{4}\right)\left(1-\frac{3X}{2}\right)(x-x_0)=0, \tag{4}$$

where $\beta=16\mu^2/243XLF$. This dispersion relation is then converted into a tunneling equation through letting

$$D^2\rightarrow\frac{c^2}{\omega^2}\frac{d^2}{dx^2}=n_\perp^2\frac{d^2}{dz^2},$$

and at the same time changing variables so that $z_0-z=\omega n_\perp x/c$, where $z_0=9\omega n_\perp L/2\mu c$. This leads to the appropriate tunneling equation, which may be written in the form

$$\psi^{vi}+(1-\epsilon)\psi^{iv}-\lambda^4z(\psi''+\psi)$$
$$=-\lambda^4z(1-1/F)(\psi''+\psi), \tag{5}$$

where ψ^{iv} and ψ^{vi} represent the fourth and sixth derivatives of ψ, respectively, and

$$\epsilon=-\frac{(1-3X/4)}{(1-3X/2)},$$

$$\lambda^4=\frac{16\mu^2(1-9X/8)}{243Xn_\perp^5}\frac{c}{\omega L},$$

$$n_\perp^2=\frac{(1-3X/4)(1-3X/2)}{(1-9X/8)},$$

$$\zeta=z/\kappa,\quad z_0=\tfrac{9}{2}\kappa,\quad \kappa=n_\perp\omega L/c\mu.$$

III. SOLVING THE TUNNELING EQUATION

The solution of this equation will proceed in two steps, where the effects of absorption are first ignored and then these effects will be included by converting the differential equation into an integral equation.

A. Solving the tunneling equation without absorption

The effects of absorption are contained in the term on the right hand side of Eq. (5) and are localized since $F\rightarrow 1$ as $|z|\rightarrow\infty$. Neglecting this term, the homogeneous equation and its adjoint (which will be needed when absorption is included) are given by

$$f^{vi}+(1-\epsilon)f^{iv}-\lambda^4z(f''+f)=0, \tag{6}$$

$$F^{vi}+(1-\epsilon)F^{iv}-\lambda^4(zF''+2F'+zF)=0. \tag{7}$$

1. The exact solutions

These equations may be solved by using the Laplace integral method, where the solution is assumed to be of the form

$$f(z)=\int_C e^{-ikz}f(k)dk,\quad F(z)=\int_C e^{-ikz}F(k)dk, \tag{8}$$

where C represents one of the six independent contours in the complex k-plane. Using these forms, Eq. (6) becomes

$$i(1-k^2)e^{-ikz}f(k)|_C+\int_C[-k^6f+(1-\epsilon)k^4f$$
$$-i\lambda^4(k^2-1)f'-2i\lambda^4kf]dk=0,$$

so that if the end points of the contours are chosen properly, and $f(k)$ satisfies the first order equation,

$$\frac{f'}{f}=\frac{ik^6-i(1-\epsilon)k^4}{\lambda^4(k^2-1)}-\frac{2k}{k^2-1}, \tag{9}$$

then the solution (and similarly, the adjoint solution) may be represented by

$$f(z)=\int_C\exp\left[-ikz+\frac{ik^5}{5\lambda^4}+\frac{i\epsilon k^3}{3\lambda^4}+\frac{i\epsilon k}{\lambda^4}\right.$$
$$\left.-\frac{i\epsilon}{2\lambda^4}\ln\left(\frac{k+1}{k-1}\right)\right]\frac{dk}{k^2-1}, \tag{10}$$

$$F(z)=\int_C\exp\left[-ikz+\frac{ik^5}{5\lambda^4}+\frac{i\epsilon k^3}{3\lambda^4}+\frac{i\epsilon k}{\lambda^4}\right.$$
$$\left.-\frac{i\epsilon}{2\lambda^4}\ln\left(\frac{k+1}{k-1}\right)\right]dk. \tag{11}$$

It follows from these integral representations that $f''+f=-F$. This representation in the k-plane has branch points at $k=\pm 1$, but if we instead write the solution in the u-plane where $k=i\tan u$, then there are no branch points in the u-plane. These tunneling equations are nearly identical to the corresponding tunneling equations for the third ion cyclotron harmonic, where, for example, the equation corresponding to Eq. (6) is[6]

$$f^{vi}+(1-\epsilon)f^{iv}-\lambda^4z(f''+f)-\gamma f=0. \tag{12}$$

For the ion case, $\gamma+\epsilon>0$, whereas for the electron case, $\gamma=0$ and $\epsilon<0$. Although this may seem a trivial change, the coupling is significantly different, requiring a renewed analysis.

2. The asymptotic solutions

Having the exact solutions, it is now possible to examine the asymptotic forms in order to obtain the scattering parameters, viz., the transmission, reflection and conversion coefficients. For this task, it is fruitful to examine the steepest

TABLE I. Saddle point parameters as $z\to\infty$.

k_0	u_0	$zh(u_0)$	$zh''(u_0)$	ϕ
$-\left(1-\frac{\lvert\epsilon\rvert}{2\lambda^4 z}\right)$	$\frac{i}{2}\ln\left(\frac{4\lambda^4 z}{\lvert\epsilon\rvert}\right)$	$iz-\frac{i\lvert\epsilon\rvert}{2\lambda^4}\ln z$	$\frac{2i\lvert\epsilon\rvert}{\lambda^4}$	$\frac{\pi}{4}$
$\left(1-\frac{\lvert\epsilon\rvert}{2\lambda^4 z}\right)$	$-\frac{i}{2}\ln\left(\frac{4\lambda^4 z}{\lvert\epsilon\rvert}\right)$	$-iz+\frac{i\lvert\epsilon\rvert}{2\lambda^4}\ln z$	$-\frac{2i\lvert\epsilon\rvert}{\lambda^4}$	$-\frac{\pi}{4}$
$i\lambda z^{1/4}$	$\frac{\pi}{2}-\frac{1}{\lambda z^{1/4}}$	$\frac{4}{5}\lambda z^{5/4}$	$-4\lambda^3 z^{7/4}$	0
$-i\lambda z^{1/4}$	$\frac{\pi}{2}+\frac{1}{\lambda z^{1/4}}$	$-\frac{4}{5}\lambda z^{5/4}$	$4\lambda^3 z^{7/4}$	$\frac{\pi}{2}$
$-\lambda z^{1/4}$	$\frac{\pi}{2}+\frac{i}{\lambda z^{1/4}}$	$\frac{4i}{5}\lambda z^{5/4}$	$4i\lambda^3 z^{7/4}$	$\frac{\pi}{4}$
$\lambda z^{1/4}$	$\frac{\pi}{2}-\frac{i}{\lambda z^{1/4}}$	$-\frac{4i}{5}\lambda z^{5/4}$	$-4i\lambda^3 z^{7/4}$	$-\frac{\pi}{4}$

descent contributions, which will correlate with the eikonal solutions for large $|z|$. The steepest descent integrals are each of the form

$$\int e^{zh(u)}du \approx \left[\frac{2\pi}{|zh''(u_0)|}\right]^{1/2} e^{zh(u_0)+i\phi},$$

where $h'(u_0)=0$ defines the saddle points u_0 and $\phi=(\pi-\psi)/2$ is the crossing angle, where $zh''(u_0)=|zh''(u_0)|e^{i\psi}$. For this case,

$$h(u)=\left(1+\frac{|\epsilon|}{\lambda^4 z}\right)\tan u-\frac{\tan^5 u}{5\lambda^4 z}-\frac{|\epsilon|\tan^3 u}{3\lambda^4 z}-\frac{|\epsilon|}{\lambda^4 z}u. \tag{13}$$

Setting $k_0=i\tan u_0$, then the condition $h'(u_0)=0$ leads to

$$-k_0^6+(1+|\epsilon|)k_0^4+\lambda^4 z(k_0^2-1)=0, \tag{14}$$

which is simply the original dispersion relation. Also, in terms of k_0, the quantity $zh''(u_0)$ is given by

$$zh''(u_0)=\frac{2ik_0(k_0^2-1)}{\lambda^4}[\lambda^4 z-3k_0^4+2(1+|\epsilon|)k_0^2]. \tag{15}$$

The asymptotic roots of Eq. (14) are

$$k_0^2\approx 1-\frac{|\epsilon|}{\lambda^4 z},\quad \text{fast waves},$$

$$k_0^2\approx \pm\lambda^2\sqrt{z},\quad \text{slow waves},$$

where $\approx$ indicates asymptotic equivalence. From these results, the location of the saddle points, the dominant terms of $zh(u_0)$ and $zh''(u_0)$ and the crossing angles are shown in Table I for $z\to\infty$ and in Table II for $z\to-\infty$.

Although the saddle point method works well for all of the slow waves, it fails to give the proper amplitude for the fast wave terms, since the paths over the fast wave saddle points decrease rapidly only on one side. The crossing angle and the phase dependence on z is correct, but an integral in the k-plane about the branch point gives a better approximation.[7] Using this result and the other elements from the tables, the asymptotic elements are

$$f_+\approx\frac{\pi e^{-\eta/2}}{\Gamma(1-i\alpha)}\exp\left[iz-i\alpha\ln\left(z+\frac{1}{\lambda^4}+\frac{9\alpha-i}{2}\right)-\frac{i}{5\lambda^4}+\frac{8i\alpha}{3}-i\alpha\ln 2\right], \tag{16}$$

$$f_-=f_+^*, \tag{17}$$

$$s_\pm\approx\frac{\sqrt{\pi}}{\sqrt{2}\lambda^{3/2}z^{7/8}}\exp\left[\pm i\left(\frac{4\lambda}{5}z^{5/4}+\frac{\pi}{4}\right)\right], \tag{18}$$

$$\sigma_+\approx\frac{\sqrt{\pi}}{\sqrt{2}\lambda^{3/2}z^{7/8}}\exp\left(\frac{4\lambda}{5}z^{5/4}\right), \tag{19}$$

$$\sigma_-\approx\frac{\sqrt{\pi}i}{\sqrt{2}\lambda^{3/2}z^{7/8}}\exp\left(-\frac{4\lambda}{5}z^{5/4}\right), \tag{20}$$

as $z\to\infty$, while for $z\to-\infty$, the fast wave terms are the same, but

$$\sigma_p\approx\frac{\sqrt{\pi}}{\sqrt{2}\lambda^{3/2}|z|^{7/8}}\exp\left(\frac{1-i}{\sqrt{2}}\frac{4\lambda}{5}|z|^{5/4}+\frac{3\pi i}{8}\right), \tag{21}$$

TABLE II. Saddle point parameters as $z\to-\infty$.

k_0	u_0	$zh(u_0)$	$zh''(u_0)$	ϕ
$-\left(1+\frac{\lvert\epsilon\rvert}{2\lambda^4\lvert z\rvert}\right)$	$\frac{\pi}{2}+\frac{i}{2}\ln\left(\frac{4\lambda^4\lvert z\rvert}{\lvert\epsilon\rvert}\right)$	$iz-\frac{i\lvert\epsilon\rvert}{2\lambda^4}\ln z$	$\frac{2i\lvert\epsilon\rvert}{\lambda^4}$	$\frac{\pi}{4}$
$\left(1+\frac{\lvert\epsilon\rvert}{2\lambda^4\lvert z\rvert}\right)$	$\frac{\pi}{2}-\frac{i}{2}\ln\left(\frac{4\lambda^4\lvert z\rvert}{\lvert\epsilon\rvert}\right)$	$-iz+\frac{i\lvert\epsilon\rvert}{2\lambda^4}\ln z$	$-\frac{2i\lvert\epsilon\rvert}{\lambda^4}$	$-\frac{\pi}{4}$
$\frac{-1+i}{\sqrt{2}}\lambda\lvert z\rvert^{1/4}$	$\frac{\pi}{2}-\frac{(1-i)}{\sqrt{2}\lambda\lvert z\rvert^{1/4}}$	$-\frac{(1+i)}{\sqrt{2}}\frac{4}{5}\lambda\lvert z\rvert^{5/4}$	$-\frac{(1-i)}{\sqrt{2}}4\lambda^3\lvert z\rvert^{7/4}$	$\frac{\pi}{8}$
$\frac{1+i}{\sqrt{2}}\lambda\lvert z\rvert^{1/4}$	$\frac{\pi}{2}-\frac{(1+i)}{\sqrt{2}\lambda\lvert z\rvert^{1/4}}$	$-\frac{(1-i)}{\sqrt{2}}\frac{4}{5}\lambda\lvert z\rvert^{5/4}$	$-\frac{(1+i)}{\sqrt{2}}4\lambda^3\lvert z\rvert^{7/4}$	$-\frac{\pi}{8}$
$\frac{-1-i}{\sqrt{2}}\lambda\lvert z\rvert^{1/4}$	$\frac{\pi}{2}+\frac{(1+i)}{\sqrt{2}\lambda\lvert z\rvert^{1/4}}$	$\frac{(1-i)}{\sqrt{2}}\frac{4}{5}\lambda\lvert z\rvert^{5/4}$	$\frac{(1+i)}{\sqrt{2}}4\lambda^3\lvert z\rvert^{7/4}$	$\frac{3\pi}{8}$
$\frac{1-i}{\sqrt{2}}\lambda\lvert z\rvert^{1/4}$	$\frac{\pi}{2}+\frac{(1-i)}{\sqrt{2}\lambda\lvert z\rvert^{1/4}}$	$\frac{(1+i)}{\sqrt{2}}\frac{4}{5}\lambda\lvert z\rvert^{5/4}$	$\frac{(1-i)}{\sqrt{2}}4\lambda^3\lvert z\rvert^{7/4}$	$-\frac{3\pi}{8}$

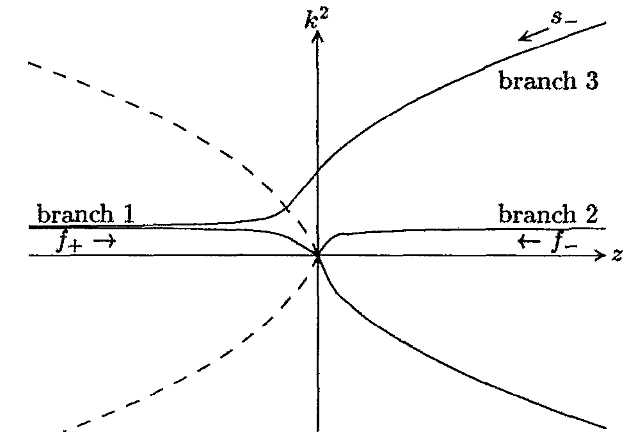

FIG. 1. Here $k^2(z)$ vs z for $|\epsilon|=2$, $\lambda^4=10$ showing the real (solid) and imaginary parts (dashed). The principal branches are indicated.

$$\sigma_m \approx \frac{\sqrt{\pi} i}{\sqrt{2}\lambda^{3/2}|z|^{7/8}} \exp\left(-\frac{1+i}{\sqrt{2}} \frac{4\lambda}{5} |z|^{5/4} + \frac{\pi i}{8} \right), \tag{22}$$

where we have introduced the tunneling factor,

$$\eta = \pi\alpha, \quad \text{with} \quad \alpha \equiv |\epsilon|/2\lambda^4. \tag{23}$$

The corresponding adjoint elements are given by

$$F_\pm \approx \frac{2\alpha}{z} f_\pm, \tag{24}$$

$$S_\pm \approx -\lambda^2 \sqrt{z} s_\pm, \tag{25}$$

$$\Sigma_\pm \approx \lambda^2 \sqrt{z} \sigma_\pm, \tag{26}$$

$$\Sigma_p \approx -i\lambda^2 \sqrt{|z|} \sigma_\pm, \tag{27}$$

$$\Sigma_m \approx i\lambda^2 \sqrt{|z|} \sigma_\pm. \tag{28}$$

At this point, it is necessary to use causality to find which elements connect together to form the physically meaningful solutions that represent an incident fast wave from the negative z side (solution f_1, which is incident on branch 1), an incident fast wave from the positive z side (solution f_2, which is incident on branch 2), and an incident slow wave (solution f_3, which is incident on branch 3). The remaining solutions may be chosen rather arbitrarily, requiring only that they make a complete set of solutions. The general character of the dispersion relation is illustrated in Fig. 1, and we note from the figure that f_+ represents a fast wave that travels from left to right, f_- represents a fast wave that travels from right to left, that s_+ represents an outgoing slow wave and s_- represents an incoming slow wave. Of the remaining elements, σ_+ and σ_p are both exponentially growing on their respective sides, and σ_- and σ_m are both exponentially decaying on their respective sides.

A sketch of the location of the saddle points and steepest descent paths in the u-plane is given in Fig. 2 for large positive z and in Fig. 3 for large negative z. It may be noted that at every odd multiple of $\pi/2$ on the real axis, the paths end, since $\tan^5 u \rightarrow \infty$ there if the approach angle is 180° or multiples of $360°/5 = \pm 72°$ from there, or at $\pm 36°$ or $\pm 108°$. From causality, it follows that for either incident fast wave

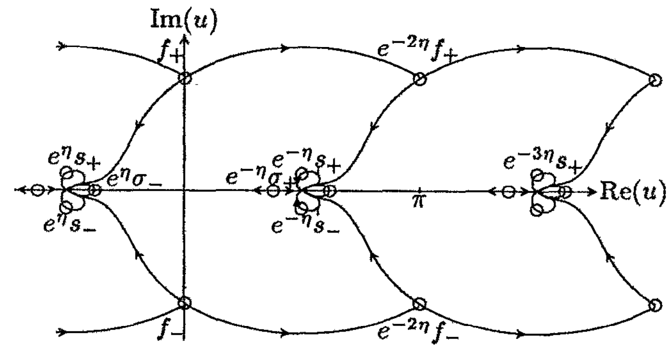

FIG. 2. The complex u-plane as $z \rightarrow \infty$, showing saddle points and acceptable paths. Each saddle point is labeled with its type and relative amplitude.

solution, no incoming slow wave is permitted, so their paths may not terminate at $-108°$ (which would require crossing the s_- saddle point), and any path ending at 180° or $\pm 36°$ must cross a growing solution saddle point for either the positive or the negative z side. Thus, the appropriate contours must start and end at 108°.

The simplest case is for solution f_1, which represents an incident wave on branch 1 of Fig. 1, so the incident wave is proportional to f_+ so the path *must* cross an f_+ saddle on the negative z side. Choosing the crossing to be the saddle labeled $e^{\eta} f_+$, the path then must start and stop at 108°, so it is reasonable to let the path start at $-\pi/2$, crossing the fast wave saddle point and terminating at $\pi/2$. This path also crosses the $e^{\eta}\sigma_m$ in the negative direction near its start and crosses the $e^{-\eta}\sigma_m$ in the positive direction near its end (the positive direction for each crossing is defined by $-\pi/2 < \phi \leq \pi/2$). Following the same contour for positive z (the path must be topologically equivalent), the path crosses the $e^{\eta} s_+$ saddle in the positive direction, the f_+ saddle in the positive sense, and then crosses the $e^{-\eta} s_+$ saddle in the negative direction. Note that the steepest descent path for each of the $f_\pm$ comes down the ridge of the neighboring saddle point, then turns 90° to continue down. This partial crossing may be ignored, since it only affects the amplitude, and the amplitude is not derived from the saddle point ap-

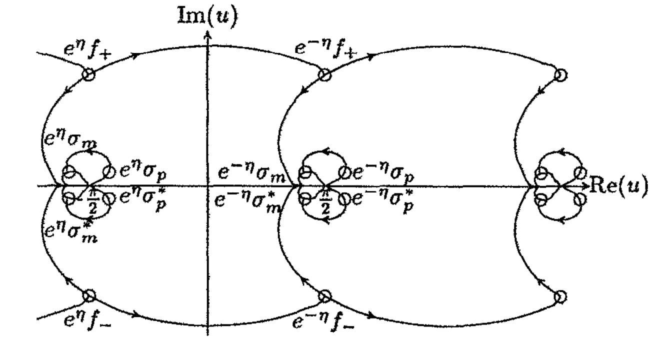

FIG. 3. The complex u-plane as $z \rightarrow -\infty$, showing saddle points and acceptable paths. Each saddle point is labeled with its type and relative amplitude.

proximation for the fast wave terms. The resulting solution may be expressed by

$$-e^{\eta}\sigma_m+e^{\eta}f_++e^{-\eta}\sigma_m\leftarrow f_1\rightarrow e^{\eta}s_++f_+-e^{-\eta}s_+, \tag{29}$$

and introducing the frequently appearing factor $\varepsilon\equiv 1-e^{-2\eta}$, this result may be written more compactly as

$$-e^{\eta}\varepsilon\sigma_m+e^{\eta}f_+\leftarrow f_1\rightarrow e^{\eta}\varepsilon s_++f_+. \tag{30}$$

This way of representing the asymptotic solutions indicates the limit on the right for large positive z and the limit on the left for large negative z. From this solution, it is clear that the amplitude transmission coefficient is $T_1=e^{-\eta}$, that the conversion coefficient is $C_1=\varepsilon$, and that there is no reflection, so $R_1=0$ (since there is no f_- term anywhere).

The solution f_2 for an incident wave on branch 2 in Fig. 1 is somewhat more complicated. The incident wave is an f_- wave on the positive z side, so the path must cross the f_- saddle point in Fig. 2. Since the path must begin and end above the axis, the path must cross the axis, presumably crossing at the $e^{\eta}\sigma_-$ and $e^{-\eta}\sigma_-$ saddle points. The simplest path to the end points from these axis crossings is for both path segments to turn to the left and cross the $e^{\eta}s_+$ and $e^{-\eta}s_+$ saddle points and terminate at 108° at $-\pi/2$ and at $\pi/2$. When this contour is examined in Fig. 3, however, this simplest path is found to be impossible, since starting at $-\pi/2$, the path *must* cross the $e^{\eta}f_+$ to get to the axis crossing between $-\pi/2$ and $\pi/2$, which involves crossing both the $e^{-\eta}\sigma_m$ and the $e^{-\eta}\sigma_m^*$ saddle points. This violates causality, since f_+ is an incoming wave from the wrong side. The proper path in Fig. 2 is to go up from the axis crossings at $e^{\eta}\sigma_-$ and $e^{-\eta}\sigma_-$ to the f_+ and $e^{-2\eta}f_+$ saddle points and then to come down and cross the $e^{-\eta}s_+$ and the $e^{-3\eta}s_+$ saddle points before terminating at $\pi/2$ and $3\pi/2$, respectively. In Fig. 3, this contour begins at $\pi/2$, crosses the $e^{-\eta}\sigma_m$ saddle point before crossing the $e^{-\eta}f_-$ saddle, and then crosses the $e^{-3\eta}\sigma_m$ saddle before ending at $3\pi/2$. Assembling the components from this contour, the f_2 asymptotic solution is

$$e^{-\eta}f_--e^{-\eta}\varepsilon\sigma_m\leftarrow f_2\rightarrow f_--\varepsilon f_++e^{-\eta}\varepsilon s_+ -e^{-\eta}\varepsilon\sigma_-. \tag{31}$$

From this result, it is apparent that $T_2=e^{-\eta}$, $R_2=-\varepsilon$, and $C_2=e^{-\eta}\varepsilon$.

For the incident slow wave solution on branch 3, f_3, the path must cross an s_- saddle in Fig. 2 for the incident wave. Starting at $-\pi/2$ at angle $-108°$, the path first crosses the $e^{\eta}s_-$ saddle. From there, it may not proceed to cross the f_- saddle, since that would be an incoming fast wave, nor may it terminate at the starting point at angle $-36°$, since that would lead to a growing solution for large negative z. It must therefore cross the axis over the $e^{\eta}\sigma_-$ saddle. From there it may cross the f_+ saddle, which represents an outgoing fast wave (converted from slow to fast), and then it may cross the $e^{-\eta}s_+$ saddle (a reflected slow wave) ending at $\pi/2$ at 108°. In Fig. 3, this contour first crosses the $e^{\eta}\sigma_m^*$ saddle, then crosses the $e^{\eta}f_-$ saddle (an outgoing fast wave on this side), then crossing the axis and the $e^{-\eta}\sigma_m$ saddle before ending at $\pi/2$ at 108°. This contour leads to the asymptotic solution,

$$-e^{\eta}\sigma_m^*+e^{-\eta}\varepsilon\sigma_m+e^{\eta}f_-\leftarrow f_3\rightarrow e^{\eta}s_--e^{-\eta}s_++f_+ +e^{\eta}\sigma_-. \tag{32}$$

From this result, it is apparent that $R_3=-e^{-2\eta}$, $C_3^+=e^{-\eta}$ and $C_3^-=1$, where $C_3^{\pm}$ refer to the conversion coefficients from the slow wave to outgoing fast waves on the positive or negative z sides.

For the remaining three independent solutions, one is chosen to contain an σ_+ element on the positive z side and the other two contain either the σ_p or the σ_p^* elements. It is also convenient to take a linear combinations of these solutions (not independent solutions), one of which is purely decaying on the positive z side, which is defined by

$$f_0\equiv f_1-f_3+e^{2\eta}(f_5-f_6), \tag{33}$$

whose asymptotic representation is

$$e^{\eta}(f_+-f_-+\sigma_p-\sigma_p^*+\sigma_m^*-\sigma_m)\leftarrow f_0\rightarrow -e^{\eta}\sigma_-, \tag{34}$$

while the other is purely decaying on the negative z side, defined by

$$f_7\equiv f_3-e^{2\eta}f_2, \tag{35}$$

whose asymptotic representation is

$$e^{\eta}(\sigma_m-\sigma_m^*)\leftarrow f_7\rightarrow e^{2\eta}(f_+-f_-)+e^{\eta}(s_--s_+) +e^{3\eta}\sigma_-. \tag{36}$$

TABLE III. Asymptotic forms for the solutions.

$z\rightarrow-\infty$	f_n	$z\rightarrow\infty$
$e^{\eta}(f_+-f_-+\sigma_p-\sigma_p^*+\sigma_m^*-\sigma_m)$	f_0	$-e^{\eta}\sigma_-$
$-e^{\eta}\varepsilon\sigma_m+e^{\eta}f_+$	f_1	$f_++e^{\eta}\varepsilon s_+$
$e^{-\eta}f_--e^{-\eta}\varepsilon\sigma_m$	f_2	$f_--\varepsilon f_++e^{-\eta}\varepsilon s_+-e^{-\eta}\varepsilon\sigma_-$
$-e^{\eta}\sigma_m^*+e^{-\eta}\sigma_m+e^{\eta}f_-$	f_3	$e^{\eta}s_--e^{-\eta}s_++f_++e^{\eta}\sigma_-$
$-e^{-\eta}\sigma_m$	f_4	$e^{-\eta}\sigma_++e^{-\eta}s_+-f_+$
$e^{-\eta}\sigma_p$	f_5	$-e^{-\eta}s_+$
$e^{-\eta}\sigma_p^*$	f_6	$-e^{-\eta}s_-$
$e^{\eta}(\sigma_m-\sigma_m^*)$	f_7	$e^{2\eta}(f_+-f_-)+e^{\eta}(s_--s_+)+e^{3\eta}\sigma_-$

The complete connection formulas are tabulated in Table III. The connection formulas are the same for the adjoint solutions by substituting upper case for lower case designations.

B. Solving the tunneling equation with absorption

Since the right hand side of Eq. (5) is localized, it is possible to treat it as a sink term and use the Green's function to find the solution, so that

$$\psi(z)=\int_{-\infty}^{\infty}G(z,y)h(y)[\psi''(y)+\psi(y)]dy, \tag{37}$$

where $h(y)=-\lambda^4y[1-1/F(y)]$ is referred to as the sink function. This is an integral equation, since ψ appears in the integrand, but both analytical and numerical results are conveniently obtained from this integral representation of the result.

C. Finding the Green's function

In order to find the form of the Green function, the integral is broken into two pieces, one where $y>z$ and the other where $y<z$. Equation (37) then becomes

$$\psi(z)=\int_{-\infty}^{z} G_-(z,y)h(y)[\psi''(y)+\psi(y)]dy + \int_{z}^{\infty} G_+(z,y)h(y)[\psi''(y)+\psi(y)]dy. \tag{38}$$

These expressions can be further modified by using the original Eq. (5) to eliminate the sink term formally, so that Eq. (38) may be written as

$$\psi(z)=\int_{-\infty}^{z} G_-(z,y)[\psi^{vi}+(1-\epsilon)\psi^{iv}-\lambda^4 y(\psi''+\psi)]dy + \int_{z}^{\infty} G_+(z,y)[\psi^{vi}+(1-\epsilon)\psi^{iv}-\lambda^4 y(\psi'' + \psi)]dy. \tag{39}$$

Integrating by parts, it may be established that this leads to

$$\begin{aligned}\psi(z)=&\int_{-\infty}^{\infty}[G^{vi}+(1-\epsilon)G^{iv}-\lambda^4(yG''+2G'+yG)]dy+[G_-,G_+]\psi^v-[G_-',G_+']\psi^{iv}+\{[G_-'',G_+'']\\&+(1-\epsilon)[G_-,G_+]\}\psi'''-\{[G_-''',G_+''']+(1-\epsilon)[G_-',G_+']\}\psi''+\{[G_-^{iv},G_+^{iv}]+(1-\epsilon)[G_-'',G_+'']\\&-\lambda^4 z[G_-,G_+]\}\psi'-\{[G_-^v,G_+^v]+(1-\epsilon)[G_-''',G_+''']+\lambda^4 z[G_-',G_+']+\lambda^4 z[G_-,G_+]\}\psi\\&+\{G\psi^v-G'\psi^{iv}+[G''+(1-\epsilon)G]\psi'''-[G'''+(1-\epsilon)G']\psi''\\&+[G^{iv}+(1-\epsilon)G''-\lambda^4 yG]\psi'-[G^v+(1-\epsilon)G'''-\lambda^4(yG'+G)]\psi\}|_{-\infty}^{\infty}.\end{aligned} \tag{40}$$

For a solution, it therefore requires that $G(y)$ satisfy the adjoint equation (or that G be a linear combination of the F_i), that G satisfy the boundary conditions at $\pm\infty$, and that $G_\pm$ satisfy the jump conditions at $y=z$, such that

$$[G_+^{(n)};G_-^{(n)}]=0, \quad n=0,1,2,3,4, \tag{41}$$

$$[G_+^v;G_-^v]=1, \tag{42}$$

where

$$[G_+^{(n)};G_-^{(n)}]\equiv \lim_{y\to z+}\frac{d^n}{dy^n}G_+(z,y)-\lim_{y\to z-}\frac{d^n}{dy^n}G_-(z,y).$$

The Green's function is constructed from the linear combinations,

$$G_+(z,y)=\sum_{i=1}^{6} A_i(z)F_i(y), \quad y>z, \tag{43}$$

$$G_-(z,y)=\sum_{i=1}^{6} B_i(z)F_i(y), \quad y<z. \tag{44}$$

The coefficients $A_i(z)$ and $B_i(z)$ are derived in the Appendix, and enable the Green's function to be written in the symmetric form

$$G(z,y)=C_0\alpha_2 f_1F_1 + C_0\begin{cases} f_2F_1-\varepsilon f_4F_0-\varepsilon f_7F_5, & y>z,\\ f_1F_2-\varepsilon f_0F_4-\varepsilon f_5F_7, & y<z,\end{cases} \tag{45}$$

where α_2 is an arbitrary constant. The integral equation, Eq. (37), now may be written in the form

$$\psi_k=f_k+f_2I_{1k}^+ + f_1I_{2k}^- - \varepsilon f_4I_{0k}^+ - \varepsilon f_0I_{4k}^- - \varepsilon f_7I_{5k}^+ - \varepsilon f_5I_{7k}^-, \tag{46}$$

where α_2 has been chosen so that $\psi_k\to f_k$ when there is no absorption $[h(y)=0]$, and the $I_{jk}^\pm$ are integrals given by

$$I_{jk}^+\equiv C_0\int_z^\infty F_j(y)\Psi_k(y)h(y)dy, \tag{47}$$

$$I_{jk}^-\equiv C_0\int_{-\infty}^z F_j(y)\Psi_k(y)h(y)dy, \tag{48}$$

where $\Psi=\psi''+\psi$ is the adjoint of ψ.

From the form of the Green's function given by Eq. (45) and the jump conditions of Eqs. (41) and (42), we may also establish the following identities among the various solutions:

$$f_2F_1^{(n)}-f_1F_2^{(n)}+\varepsilon[f_0F_4^{(n)}-f_4F_0^{(n)}+f_5F_7^{(n)}-f_7F_5^{(n)}]=0, \quad n=0,1,2,3,4, \tag{49}$$

$$f_2F_1^v-f_1F_2^v+\varepsilon[f_0F_4^v-f_4F_0^v+f_5F_7^v-f_7F_5^v]=-2\pi i\lambda^4\varepsilon. \tag{50}$$

Although the expressions for the D_n given in Eqs. (A4)–(A9) were originally derived asymptotically, these identities may be used to prove that they are exact.

TABLE IV. Amplitude and energy scattering parameters.

Transmission	$T_1=e^{-\eta}(1+I_{21})$ $\lvert T_1\rvert^2=e^{-2\eta}\lvert 1+I_{21}\rvert^2$	$T_2=e^{-\eta}(1+I_{12})$ $\lvert T_2\rvert^2=e^{-2\eta}\lvert 1+I_{12}\rvert^2$	$T_3=0$ $\lvert T_3\rvert^2=0$
Reflection	$R_1=e^{-2\eta}I_{11}$ $\lvert R_1\rvert^2=e^{-4\eta}\lvert I_{11}\rvert^2$	$R_2=-\varepsilon+I_{22}$ $\lvert R_2\rvert^2=\lvert\varepsilon-I_{22}\rvert^2$	$R_3=-e^{-2\eta}(1-\varepsilon I_{33})$ $\lvert R_3\rvert^2=e^{-4\eta}\lvert 1-\varepsilon I_{33}\rvert^2$
Conversion	$C_1=\varepsilon(1+e^{-2\eta}I_{31})$ $\lvert C_{13}\rvert^2=\varepsilon\lvert 1+e^{-2\eta}I_{31}\rvert^2$	$C_2=e^{-\eta}\varepsilon(1+I_{32})$ $\lvert C_{23}\rvert^2=e^{-2\eta}\varepsilon\lvert 1+I_{32}\rvert^2$	$C_3^+=e^{-\eta}(1+I_{23})$ $\lvert C_{32}\rvert^2=e^{-2\eta}\varepsilon\lvert 1+I_{23}\rvert^2$ $C_3^-=(1+e^{-\eta}I_{13})$ $\lvert C_{31}\rvert^2=\varepsilon\lvert 1+e^{-2\eta}I_{13}\rvert^2$

D. Solving for the scattering parameters

By examining the asymptotic form of Eq. (46), it is possible to obtain revised scattering parameters, which include the effects of absorption. It must first be noted that as $z\to\infty$, all of the I_{jk}^+ integrals vanish, as they no longer have any finite range. One must look more closely at the I_{4k}^- integral, however, since it has an exponentially growing integrand, and the corresponding integral grows at the same rate. This term is, however, multiplied by f_0, which is exponentially decaying, so in the limit, the product vanishes as an inverse power of z. In this limit, then we have

$$\psi_k(z)\to f_k(z)+f_1I_{2k}-\varepsilon f_5I_{7k}, \quad \text{as } z\to\infty. \tag{51}$$

Strictly speaking, this expression is not valid for $k=4$, since f_4 itself is exponentially growing, and there are some convergence difficulties. The integrals in this expression have no superscript, as they are now over the infinite range $[-\infty,\infty]$.

In the opposite limit, as $z\to-\infty$, the integrals I_{jk}^- vanish, and again the product of a decaying function times a growing integral vanishes, so both the $f_4I_{0k}^+$ term and the $f_7I_{5k}^+$ term vanish. For this limit, then, the result is

$$\psi_k(z)\to f_k(z)+f_2I_{1k}, \quad \text{as } z\to-\infty. \tag{52}$$

Again, this expression is invalid for $k=5,6$, where f_5 and f_6 are growing.

If these two asymptotic expressions are now written in terms of the asymptotic forms of Table III, then the principal solutions may be represented by

$$e^{\eta}f_++e^{-\eta}I_{11}f_-\leftarrow\psi_1\to f_+(1+I_{21}) + e^{\eta}\varepsilon(1+e^{-2\eta}I_{31})s_+, \tag{53}$$

$$e^{-\eta}(1+I_{12})f_-\leftarrow\psi_2\to f_--(\varepsilon-I_{22})f_+ + e^{-\eta}\varepsilon(1+I_{32})s_+, \tag{54}$$

$$e^{\eta}(1+e^{-2\eta}I_{13})f_-\leftarrow\psi_3\to e^{\eta}s_-+(1+I_{23})f_+ - e^{-\eta}(1-\varepsilon I_{33})s_+, \tag{55}$$

from which we can extract the revised scattering parameters. The amplitude and energy scattering parameters are given in Table IV, where the difference in amplitudes between the $f_\pm$ and the $s_\pm$ mean that while the transmission and reflection coefficients are simply the magnitude squared of the amplitude coefficients, there is an extra coefficient between fast and slow conversions, so that, for example, the conversion coefficient between branch 1 and branch 3 is $|C_{13}|^2=|C_1|^2/\varepsilon$. It may be proved that $I_{jk}=I_{kj}$, which then leads to reciprocity since then $|C_{13}|^2=|C_{31}|^2$ and $|C_{23}|^2=|C_{32}|^2$, which means that the fraction of the converted energy between fast and slow waves is the same in either direction. The reciprocity between the two transmission coefficients is even more trivial, since by examining the integrals I_{11}, I_{12} and I_{21} in the complex z-plane,[9] it may be proved that each of these integrals vanish identically, so that $T_1=T_2=T=e^{-\eta}$, and $R_1=0$. The other integrals are not so trivial.

IV. SYNCHROTRON EMISSION

The previous section has dealt with absorption formally, and exactly without approximation, but except for those integrals that vanish identically, no exact results are yet available to determine the absorption. In order to accurately assess the amount of absorption on each branch, the remaining I_{jk} integrals must be evaluated numerically. This is a somewhat onerous task, but straightforward numerical procedures described elsewhere[10] allow one to obtain complete solutions for all three physically meaningful solutions over a broad range of parameters.

For the emission, however, the problem is simpler, since only I_{22} is required, and this can be obtained with high accuracy without solving the integral equation. The keys to this implementation are first that the emission on branches 1 and 2 is given by[1]

$$E_1=(1-e^{-2\eta})I_{BB}, \tag{56}$$

$$E_2=(1-e^{-2\eta}-|R_2|^2)I_{BB}, \tag{57}$$

where I_{BB} is the blackbody emission intensity, and second, that a new asymptotic method has been developed for evaluating $|R_2|^2=|\varepsilon-I_{22}|^2$ (see Table IV), which requires only a simple summing of a series.[2] Branch 1 corresponds to radiation on the high magnetic field side, which is given by the classical expression for synchrotron radiation. Branch 2 is the low magnetic field side, and here the reflection term is important in certain parameter ranges, and emission on this branch is the most commonly used. The remainder of this paper is devoted to evaluating I_{22} and developing simple but accurate empirical formulas for $|R_2|$.

That the emission should be described by Eqs. (56) and (57) above is not immediately transparent, since the full expressions for the two cases can be represented by the single formula,

$$E_k=(1-|T_k|^2-|R_k|^2-|C_{k3}|^2)I_{BB}. \tag{58}$$

It has been proved analytically, however, that $T_1=e^{-\eta}$ and $R_1=0$. Furthermore, if one assumes that the mode converted energy (represented by $|C_{k3}|^2$ for the fraction of incident power converted from branch k to branch 3, the slow wave) is totally absorbed sufficiently close to the cyclotron layer (supported by ray tracing studies), so that the absorption region is characterized by the same temperature as the mode conversion region, then the slow wave absorption region is a blackbody radiator of the slow wave, which is converted back and adds to the emission. But from reciprocity, the fraction converted back is exactly the same and the conversion term cancels out, leaving one with the simplified expressions above. It is ironic that Eq. (56), which is a function of η only, which in turn is completely independent of absorption, should describe absorption, but providing the slow wave is absorbed nearby, the classical expression is exact. From the low field side, however, the reflection is nonvanishing, and evaluating this correction to the classical formula forms the principal subject of this paper.

A. Evaluating I_{22}

For this case, the integral is given by

$$I_{22}=\frac{-1}{2\pi i\lambda^4\varepsilon}\int_{-\infty}^{\infty}F_2(z)\Psi_2(z)h(z)dz, \tag{59}$$

where the localized absorption term is given by

$$h(z)=-\lambda^4 z(1-1/F)=-\lambda^4\kappa[\zeta-1/F_{9/2}(\zeta-\tfrac{9}{2})], \tag{60}$$

with $\zeta=z/\kappa$. The technique for evaluating this integral is to expand each of the three terms in the integrand in an asymptotic expansion in $1/z$, deform the path so that it is comprised of a circular path of infinite radius (so that the intrinsic divergence of the asymptotic series for any finite radius is avoided), and a piece that dips back and circles the origin, picking up each pole. This results in a power series in κ^2, which is convergent within some finite radius of convergence (typically, $\kappa_{\max}\sim 0.4$).

The first step is to develop the series for $h(z)$, which is given by

$$h(z)\to\sum_{n=1}^{\infty}\frac{h_n}{z^n}. \tag{61}$$

To find the coefficients h_n, the series for $F_{9/2}$, whose asymptotic series is given by

$$F_q(\zeta)\to\frac{1}{\zeta}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(n+q)}{\Gamma(q)\zeta^n}, \tag{62}$$

must be inverted. Since the function needed is $F_q(\zeta-q)$, a shift is first required, given by the series

$$F_q(\zeta-q)\to\sum_{n=1}^{\infty}\frac{A_n(q)}{\zeta^n}, \tag{63}$$

where the $A_n(q)$ may be found from Eq. (62) to be

$$A_n(q)=-\frac{1}{\Gamma(q)}\sum_{m=1}^{n}(-1)^m\Gamma(q+m-1)C_{n-1}^{n-m}q^{n-m}, \tag{64}$$

where $C_m^n=n!/m!(n-m)!$ are the binomial coefficients. The first few values are $A_1=1$, $A_2=0$, and $A_3=q=\frac{9}{2}$ for this case. The inversion is then accomplished by the expression

$$\frac{1}{F_q(\zeta-q)}=\zeta\left[1+\sum_{n=2}^{\infty}\frac{C_n(q)}{\zeta^n}\right], \tag{65}$$

where

$$C_{2m+1/2\pm1/2}(q)=\sum_{n=1}^{m}[-A_3(q)]^n \times D_{2(m-n)+1/2\pm1/2}(q,m),$$

$$D_n(q,m)=\sum_{k=0}^{n}D_{n-k}(q,m-1)D_k(q,1),$$

$$D_n(q,1)=A_{n+3}(q)/A_3(q).$$

From these expressions, it follows that the h_n coefficients of Eq. (61) are given by

$$h_n=-\lambda^4\kappa^{n+1}C_{n+1}(9/2). \tag{66}$$

The first few of these are $C_2=-q=-9/2$, $C_3=2q=9$, and $C_4=-2q(q+3)=-135/2$.

The next step is to find the expansions for F_2 and Ψ_2, each of which is dominated by their fast wave terms, so that the expansions required are of the form

$$F_-(z)=c_-\sum_{n=1}^{\infty}\frac{a_n}{z^n}e^{-i(z-\alpha\ln z)}, \tag{67}$$

$$\Psi_-(z)=c_-\sum_{n=1}^{\infty}\frac{\tilde{a}_n}{z^n}e^{-i(z-\alpha\ln z)}, \tag{68}$$

where

$$c_-=\frac{2\pi i}{\Gamma(i\alpha)}\exp\left[\frac{\eta}{2}+i\left(\frac{1}{5\lambda^4}-\frac{8\alpha}{3}+\alpha\ln 2\right)\right]. \tag{69}$$

The coefficients are determined from Eqs. (6) for f_- and (5) for ψ_- along with the adjoint relations

$$F=f''+f, \tag{70}$$

$$\Psi=\psi''+\psi. \tag{71}$$

Denoting the mth derivatives of these functions as

$$f_-^{(m)}(z)=c_-\sum_{n=0}^{\infty}(-i)^m\frac{a_{n,m}}{z^n}e^{-i(z-\alpha\ln z)}, \tag{72}$$

$$\psi_-^{(m)}(z)=c_-\sum_{n=0}^{\infty}(-i)^m\frac{\tilde{a}_{n,m}}{z^n}e^{-i(z-\alpha\ln z)}, \tag{73}$$

the coefficients for the derivatives may be determined in terms of the zeroth order coefficients through the recursion relation

$$a_{n,m}=a_{n,m-1}-[\alpha+(n-1)i]a_{n-1,m-1}, \tag{74}$$

for either the $a_{n,m}$ or the $\tilde{a}_{n,m}$. In each case, $a_{0,m}=\tilde{a}_{0,m}\equiv 1$. The remaining zero order coefficients are then found by substituting into the differential equations, and setting the coefficients of each power of z^{-n} to zero, resulting in

$$\begin{aligned}a_{n,0}=&\frac{\alpha+(n-1)i}{2i\lambda^4 n}[a_{n-1,5}+a_{n-1,4}\\&-|\epsilon|(a_{n-1,3}+a_{n-1,2}+a_{n-1,1})\\&-\lambda^4(\alpha-ni)a_{n-1,0}],\end{aligned} \tag{75}$$

$$\begin{aligned}\tilde{a}_{n,0}=&\frac{1}{2i\lambda^4 n}\Bigg\{[\alpha+(n-1)i][\tilde{a}_{n-1,5}+\tilde{a}_{n-1,4}\\&-|\epsilon|(\tilde{a}_{n-1,3}+\tilde{a}_{n-1,2}+\tilde{a}_{n-1,1})\\&-\lambda^4(\alpha-ni)\tilde{a}_{n-1,0}]-\sum_{m+1}^{n-1}h_{n-m}\tilde{\alpha}_n\Bigg\},\end{aligned} \tag{76}$$

where it is clear that the $\tilde{a}_{n,m}$ involve absorption through the h_n while the $a_{n,m}$ do not, and

$$\alpha_n=a_{n,0}-a_{n,2}=[\alpha+(n-1)i](a_{n-1,1}+a_{n-1,0}), \tag{77}$$

with a corresponding relation for $\tilde{\alpha}$.

The final step is to do the integral, which is of the form

$$I_{22}=\frac{-c_-^2}{2\pi i\lambda^4\varepsilon}\int\sum_{n=3}^{\infty}\frac{\gamma_n}{z^n}e^{-2i(z-\alpha\ln z)}dz, \tag{78}$$

where the coefficients are obtained from the triple product, so that

$$\gamma_n=\sum_{k=1}^{n-2}h_k\sum_{\ell=1}^{n-k-1}a_\ell^{(0)}\tilde{a}_{n-k-\ell}^{(0)}. \tag{79}$$

The integral required is

$$\int_{-\infty}^{\infty}z^{-n}e^{-2i(z-\alpha\ln z)}dz=\frac{e^{-(n-2i\alpha)\ln(i/2)}}{2i}\times\int_C(-t)^{-(n-2i\alpha)}e^{-t}dt,$$

where $2iz\to t$ and the contour now follows a semicircular arc from $-\infty$ to $-i\infty$, where the contribution vanishes, then goes up on the left side of the imaginary axis until it circles the origin (which is a branch point with the branch line extending downward in the complex z-plane) and comes back to $-i\infty$ to the right of the imaginary axis, and then goes to $+\infty$ along another semicircular arc, where again there is no contribution. The contribution around the branch cut is given by the Hankel integral, so that the final answer is

$$I_{22}=\frac{i}{2\lambda^4\varepsilon}\left[\frac{\pi e^{-\eta}\exp(i/5\lambda^4-8i\alpha/3)}{\alpha\Gamma(-i\alpha)}\right]^2\sum_{n=3}^{\infty}\frac{(-2i)^n\gamma_n}{\Gamma(n-2i\alpha)}. \tag{80}$$

The results for the sum, and for the reflection coefficient can be represented in the form

$$R_2=R_{20}e^{-9q\kappa^2}, \tag{81}$$

where $R_{20}=-\varepsilon$. The quantity q thus defined is a function of the dimensionless parameters κ, $X=\omega_{pe}^2/\omega^2$, and $\ell=\omega L/c$. The expressions for q are obtained from the summation of Eq. (80), but while the series is convergent over a broad range of plasma parameters, it has a finite radius of convergence near $\kappa\sim0.42$. In this range, typically $0.7<q\leqslant 1$. Empirical formulas that give values of q that are generally accurate to within one percent are given by

$$\begin{aligned}&q(\kappa,X,\ell)=1-1.926(1-e^{-a(X,\ell)\kappa^2})/a(X,\ell),\\&a(X,\ell)=4+[b(\ell)X+c(\ell)X^2]\exp[-1/(72-108X)],\\&b(\ell)=0.127-302.4/\ell,\\&c(\ell)=-0.1581-472.1/\ell+10200/\ell^2.\end{aligned} \tag{82}$$

As $X\to 2/3$ (the R-wave cutoff), these semi-empirical formulas fail, but are valid up to 97% of this limit. The parameter $a(X,\ell)$ has a minimum near $X=0.59$, and for $\ell<63.4$, $a<0$ and again the formulas are invalid. The range of validity is thus within the bounds $0<\kappa<0.42$, $0<X<0.64$, and $64<\ell$, although it is expected that the limit on κ may be extended, but the upper bound of validity is not known.

V. CONCLUSIONS

The final result of this analysis is an accurate and readily obtained expression for the reflection coefficient from the low field side, along with an extension of a previous proof, which shows that the reflection from the high field side vanishes identically. Both the accuracy and speed of this method surpasses that of any other known numerical methods for solving generalized tunneling differential equations.

For practical purposes, the principal concern is to estimate how much error might occur in measuring the electron temperature from an absolute measurement of the synchrotron radiation. If one considers the error due to the neglect of the reflection coefficient, the percentage error from Eq. (57) can be represented by

$$\%\ \text{Error}=\frac{|R_2|^2}{1-e^{-2\eta}-|R_2|^2}\times 100. \tag{83}$$

From this expression, the maximum error seems always to occur for $X=0.592$, and to a good approximation, this maximum error is expressed in terms of ℓ by the relations

$$\% \text{ Error} \approx 80/\ell, \quad @T_e \approx 3.3\times 10^5/\ell, \tag{84}$$

where T_e is in eV, and represents the temperature where the maximum error occurs for a given value of ℓ. This upper bound is generally small, giving a maximum error of 1.6% for $\ell=50$ (even though the empirical formulas fail for $\ell<64$, the series is valid) at $T_e=6.7$ keV, and this value of ℓ corresponds to approximately $L/\lambda\sim 3$. For even smaller values of ℓ (and larger errors), the slowly varying approximation upon which the entire analysis is based begins to fail. This means that the emission at the third electron cyclotron harmonic will nearly always be within one percent of that given by the classical formula where reflection is neglected. For most laboratory plasmas, even $\ell=50$ is unrealistically low (probably at least by an order of magnitude), but for magnetospheric or space plasmas, such parameters are possible. Even there, however, the effect is small until the scale length begins to approach the wavelength of the X-mode.

Although other estimates of the reflection coefficient at $3\omega_{ce}$ are generally lacking in the literature, a simple extrapolation of one estimate gives some basis for comparison. One expression for the reflection of the O-mode at the fundamental gives[11]

$$R_2=R_{20}(1-i\chi)^{-7/2}, \tag{85}$$

where the 7/2 is the order of the weak relativistic dispersion function and $\chi\approx\kappa$. With small κ, this leads to an expression of the form

$$|R_2|^2\approx R_{20}^2 \exp[-\tfrac{7}{2}\kappa^2(1-\tfrac{1}{2}\kappa^2+\cdots)]. \tag{86}$$

Changing the 7/2 to 9/2 for the third harmonic X-mode, there is still a basic factor of 2 difference in the exponent, which predicts a larger reflection coefficient. This calculation was not based on a tunneling equation and hence neglected mode conversion, but it does have similar character for the limiting case of small κ.

The most important result is that within the approximations leading the tunneling equation with absorption of Eq. (5), the series expression is exact, and the semi-empirical formulas that approximate the results to high accuracy lead to reliable error estimates in ECE emission if reflection is neglected, and permit even higher accuracy if the reflection is included.

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APPENDIX: DERIVATION OF THE GREEN'S FUNCTION COEFFICIENTS

It is required that $B_5=B_6=0$ in Eq. (44), since the boundary condition requires a bounded solution as $y\to-\infty$, and $F_5(y)$ and $F_6(y)$ contain σ_p or σ_p^* terms. For similar reasons, $A_4=0$ in Eq. (43) since $F_4(y)$ contains a σ_+ term as $y\to\infty$. For the specific solution ψ_1, the Green's function cannot permit an incoming fast wave term, f_-, as $y\to\infty$, so $A_2=0$. For this same solution, no incoming slow wave is allowed, so $A_6=e^{2\eta}A_3$. This leaves seven constants with six equations [Eqs. (41) and (42)], so one constant is simply an arbitrary amplitude. This may be represented by letting one of the constants be dependent on the others, and one choice is to let $B_1=\alpha_2B_2+\alpha_3B_3+\alpha_4B_4$, with the α's to be determined. The jump conditions can then be used to determine the remaining constants through the linear set,

$$\begin{pmatrix} F_1 & F_3+e^{2\eta}F_6 & F_5 & -(F_2+\alpha_2F_1) & -(F_3+\alpha_3F_1) & -(F_4+\alpha_4F_1) \\ F_1' & F_3'+e^{2\eta}F_6' & F_5' & -(F_2'+\alpha_2F_1') & -(F_3'+\alpha_3F_1') & -(F_4'+\alpha_4F_1') \\ F_1'' & F_3'+e^{2\eta}F_6'' & F_5'' & -(F_2''+\alpha_2F_1'') & -(F_3''+\alpha\_3F_1'') & -(F_4''+\alpha_4F_1'') \\ F_1''' & F_3'+e^{2\eta}F_6''' & F_5''' & -(F_2'''+\alpha_2F_1''') & -(F_3'''+\alpha_3F_1''') & -(F_4'''+\alpha_4F_1''') \\ F_1^{iv} & F_3^{iv}+e^{2\eta}F_6^{iv} & F_5^{iv} & -(F_2^{iv}+\alpha_2F_1^{iv}) & -(F_3^{iv}+\alpha_3F_1^{iv}) & -(F_4^{iv}+\alpha_4F_1^{iv}) \\ F_1^{v} & F_3^{v}+e^{2\eta}F_6^{v} & F_5^{v} & -(F_2^{v}+\alpha_2F_1^{v}) & -(F_3^{v}+\alpha_3F_1^{v}) & -(F_4^{v}+\alpha_4F_1^{v}) \end{pmatrix} \begin{pmatrix} A_1 \\ A_3 \\ A_5 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{A1}$$

The determinant of coefficients in Eq. (A1) is related to the Wronskian, defined by

$$W\equiv\begin{vmatrix} F_1 & F_2 & \cdots & F_6 \\ F_1' & F_2' & \cdots & F_6' \\ \vdots & \vdots & & \vdots \\ F_1^{v} & F_2^{v} & \cdots & F_6^{v} \end{vmatrix} = -(2\pi i\lambda^4)^3\varepsilon|\epsilon|. \tag{A2}$$

In terms of the Wronskian, each of the coefficients is a linear combination of the 5×5 determinants, defined by

$$D_k\equiv\frac{1}{W}\begin{vmatrix} F_1 & \cdots & F_{k-1} & F_{k+1} & \cdots & F_6 \\ F_1' & \cdots & F_{k-1}' & F_{k+1}' & \cdots & F_6' \\ \vdots & & \vdots & \vdots & & \vdots \\ F_1^{iv} & \cdots & F_{k-1}^{iv} & F_{k+1}^{iv} & \cdots & F_6^{iv} \end{vmatrix}, \tag{A3}$$

so that k denotes the missing column. Then these determinants are

$$D_1 = \frac{-e^{-2\eta}}{2\pi i\lambda^4}\left[\frac{f_2}{\varepsilon} - f_4\right], \tag{A4}$$

$$D_2 = \frac{-1}{2\pi i\lambda^4}\left[\frac{e^{-2\eta}f_1}{\varepsilon} + f_5\right], \tag{A5}$$

$$D_3 = \frac{-e^{-2\eta}}{2\pi i\lambda^4}(f_4 + f_5), \tag{A6}$$

$$D_4 = \frac{1}{2\pi i\lambda^4}[e^{-2\eta}(f_1 - f_3) + f_5 - f_6] = \frac{e^{-2\eta}f_0}{2\pi i\lambda^4}, \tag{A7}$$

$$D_5 = \frac{1}{2\pi i\lambda^4}(e^{-2\eta}f_7 + f_4), \tag{A8}$$

$$D_6 = \frac{f_4}{2\pi i\lambda^4}. \tag{A9}$$

The coefficients are then given by

$$A_1 = e^{2\eta}D_1 + e^{2\eta}\alpha_2 D_2 - \alpha_3(e^{2\eta}D_3 + D_6) + e^{2\eta}\alpha_4 D_4, \tag{A10}$$

$$A_3 = -D_6, \tag{A11}$$

$$A_5 = e^{2\eta}D_5, \tag{A12}$$

$$B_1 = e^{2\eta}\alpha_2 D_2 - \alpha_3(e^{2\eta}D_3 + D_6) + e^{2\eta}\alpha\_4 D_4, \tag{A13}$$

$$B_2 = e^{2\eta}D_2, \tag{A14}$$

$$B_3 = -(e^{2\eta}D_3 + D_6), \tag{A15}$$

$$B_4 = e^{2\eta}D_4, \tag{A16}$$

so that the final values are

$$2\pi i\lambda^4 A_1 = -f_2/\varepsilon + f_4 - \alpha_2(f_1/\varepsilon + e^{2\eta}f_5) + \alpha_3 f_5 + \alpha_4 f_0, \tag{A17}$$

$$2\pi i\lambda^4 A_3 = -f_4, \tag{A18}$$

$$2\pi i\lambda^4 A_5 = f_7 + e^{2\eta}f_4, \tag{A19}$$

$$2\pi i\lambda^4 B_1 = -\alpha_2(f_1/\varepsilon + e^{2\eta}f_5) + \alpha_3 f_5 + \alpha_4 f_0, \tag{A20}$$

$$2\pi i\lambda^4 B_2 = -f_1/\varepsilon - e^{2\eta}f_5, \tag{A21}$$

$$2\pi i\lambda^4 B_3 = f_5, \tag{A22}$$

$$2\pi i\lambda^4 B_4 = f_0. \tag{A23}$$

Assembling the various components, the Green's function can then be written as

$$G(z,y) = C_0\alpha_2 f_1 F_1 + C_0\begin{cases} f_2F_1 - \varepsilon f_4F_0 - \varepsilon f_7F_5 - \varepsilon(\alpha_3 - \alpha_2 e^{2\eta})f_5F_1 - \varepsilon\alpha_4 f_0F_1, & y>z, \\ f_1F_2 - \varepsilon f_0F_4 - \varepsilon f_5F_7 - \varepsilon(\alpha_3 - \alpha_2 e^{2\eta})f_5F_1 - \varepsilon\alpha_4 f_0F_1, & y<z, \end{cases} \tag{A24}$$

where $C_0 = -(2\pi i\lambda^4\varepsilon)^{-1}$ and each combination is of the form $f_i(z)F_j(y)$. It may now be seen that $\alpha_4 = 0$, since an $f_0(z)$ term is not permitted as $z \to -\infty$ since it is a growing term. It is also required that $\alpha_3 = e^{2\eta}\alpha_2$ to eliminate the $f_5(z)$ term as $z \to -\infty$, since it is also a growing term. With these changes, the result is

$$G(z,y) = C_0\alpha_2 f_1 F_1 + C_0\begin{cases} f_2F_1 - \varepsilon f_4F_0 - \varepsilon f_7F_5, & y>z, \\ f_1F_2 - \varepsilon f_0F_4 - \varepsilon f_5F_7, & y<z, \end{cases} \tag{A25}$$

and now α_2 is an arbitrary constant.

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