Spectral-element adaptive refinement magnetohydrodynamic simulations of the island coalescence instability

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Abstract

A recently developed spectral-element adaptive refinement incompressible magnetohydrodynamic (MHD) code is applied to simulate the problem of island coalescence instability in two dimensions. The MHD solver is explicit, and uses the Elsasser formulation [Elsasser, Phys. Rev. 79, 183 (1950)] on high-order elements. It automatically takes advantage of the adaptive grid mechanics that have been described elsewhere [Rosenberg, Fournier, Fischer, Pouquet, J. Comp. Phys. 215, 59-80 (2006)], allowing both statically refined and dynamically refined grids. The island coalescence instability is a fundamental MHD process that can produce strong current sheets and subsequent reconnection and heating in a high-Lundquist number plasma such as the solar corona [cf., Ng and Bhattacharjee, Phys. Plasmas, 5, 4028 (1998)]. Due to the formation of thin current sheets, it is highly desirable to use statically or adaptively refined grids to resolve them, and to maintain accuracy at the same time. The outputs of the spectral-element static refinement simulations are compared with simulations using finite difference method with the same refinement grid, as well as pseudo-spectral simulations with a grid.

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Introduction

◊ In some problems, such as the Parker’s model of solar corona heating by current sheet formation, structures with very sharp gradient (or even singular) develop so that numerical simulations using uniform grid is almost impossible. It is then important to try to use non-uniform grid with or static refinement.

◊ Whether truly singular current sheets (tangential discontinuities) will form in the Parker’s model is still an unresolved question. While our recent analytic work [Ng and Bhattacharjee 1998] shows that this is the case, some have argued that only smooth (although with very large gradient) current layers will form. To try to distinguish these two scenarios by simulations, one not only needs to have enough resolution to resolve the current layers, but also high accuracy.

◊ Usual (pseudo) spectral method is of high accuracy, especially in higher order derivatives, but is based on uniform grid. On the other hand, applying non-uniform grids with finite differencing may suffer more in accuracy. The spectral element method with spatial ($h$-) adaptive refinement may provide a middle ground to combine accuracy and adaptability.

◊ A spectral element code, the Geophysical-astrophysical spectral-element adaptive refinement (GASpAR) code, has been developed [Rosenberg et al. 2006]. We apply this code to the problem of island coalescence instability, which can be viewed as a 2D version of the Parker’s model. The results are compared with those from an AMR finite difference code with the same grid, and from a pseudo-spectral code with its accuracy checked by over-resolved runs.
**GASpAR** [Rosenberg et al. 2006]  

The spectral-element method uses a Galerkin variational weighted-residual approach where the equations are cast into a residual-function form, multiplied by test functions, and the integrated residual is set to zero. The test functions are also the basis functions in which the variables are expanded. Our treatment discretizes on nonconforming grids (element interfaces may not line up geometrically), but still requires the function to be continuous across interfaces.

![Diagram of mortar structures](image)

Schematic of mortar structures that represent global degrees of freedom (d.o.f.). Fields are interpolated to/from these mortar d.o.f. from/to the child elements to maintain $C^0$ continuity.
GASpAR: Staggered grid

Refinement decomposes a given element into four equal-size children. Elements are tagged for refinement or coarsening based on several criteria: a spectral estimator; tolerances on first and second derivatives; or any combination. For Navier-Stokes and magnetohydrodynamics (MHD), a staggered grid is used, where the velocity and magnetic field reside on Gauss-Lobatto, and the pressure (or vorticity) resides on Gauss nodes.
**GASpAR: MHD**

The magnetohydrodynamics (MHD) solver solves the incompressible MHD equations in Elsasser form. This solver currently allows only an explicit time stepping scheme.

Snapshots of current-density diagnostic for evolution of the Orszag-Tang vortices in MHD. Red is high and blue is low. Note the very thin, strong current sheet at the center. The grid adapts dynamically on the velocity and magnetic fields (not current density) using the spectral estimator. ([Rosenberg, Fournier, and Pouquet 2006])
2D island coalescence instability: initial condition

2D incompressible MHD equations:

$$\frac{\partial \mathbf{A}}{\partial t} + [\phi, \mathbf{A}] = [A, J] + \nu \nabla^2 \mathbf{A}$$

$$\frac{\partial \phi}{\partial t} + [\phi, A] = \eta \nabla^2 \phi$$

where $\mathbf{B} = \nabla \times \mathbf{A}$ is the magnetic field, $\mathbf{v} = \nabla \times \mathbf{A}$ is the fluid velocity, $\Omega = -\nabla^2 \phi$ is the $z$-component of the vorticity, $J = -\nabla^2 A$ is the $z$-component of the current density, $[\phi, A] \equiv \phi_y A_x - \phi_x A_y$, $\eta$ is the resistivity, and $\nu$ is the viscosity.

**Initial conditions**

The initial equilibrium (with periodic b. c.) is chosen to be:

$$A(x, y, t = 0) = A_0 \sin(2\pi x) \sin(2\pi y), \text{ with } A_0 = 0.4.$$  

To start the instability, a small initial flow is added:

$$\phi(x, y, t = 0) = \phi_0 [\cos(2\pi x) - \cos(2\pi y)], \text{ with } \phi_0 = 0.002.$$
2D island coalescence instability: time evolution

$t = 0.6$

$t = 0.8$

$t = 1.0$

$\eta = \nu = 0.0001, 1024^2$, spectral run.
2D island coalescence instability: time evolution

\( t = 0.6 \)

\( t = 0.8 \)

\( t = 1.0 \)

\( \eta = \nu = 0.0001, \ 1024^2, \ \text{spectral run.} \)
**GASpAR grid** ($\eta = \nu = 0.002, 128^2$)

$p = 8$ in GASpAR

d.o.f. = 15872

Dynamic refinement is turned off so as to compare with the finite difference code.

AMR finite difference code uses the same grid with each square having an 8×8 uniform grid within.

Pseudo spectral code uses a uniform grid.

d.o.f. = 32768
Time series comparisons \((\eta = \nu = 0.002, 128^2)\)

Kinetic energy
\[
E_K = \frac{1}{2} \int_0^1 dx \int_0^1 dy |\nabla \phi|^2
\]
black dashed – spectral
red dashed – GASpAR
light blue dashed – FD

Total energy: \(E_M + E_K\)
black solid – spectral
red solid – GASpAR
light blue solid – FD

where
\[
E_M = \frac{1}{2} \int_0^1 dx \int_0^1 dy |\nabla A|^2
\]
is the Magnetic energy.

Note that the light blue solid curve nearly covers the red and black solid curves; and the red dashed curve covers the black dashed curve.
**Time series comparisons** \( (\eta = \nu = 0.002, 128^2) \)

\[
\langle J^2 \rangle = \int_0^1 dx \int_0^1 dy J^2
\]

black solid – spectral
red solid – GASpAR
light blue solid – FD

\[
\langle \Omega^2 \rangle = \int_0^1 dx \int_0^1 dy \Omega^2
\]

black dashed – spectral
green solid – GASpAR
blue solid – FD

Note: the red curve nearly covers the black solid, and the green covers the black dashed.
**Time series comparisons** \((\eta = \nu = 0.002, 128^2)\)

Maximum current density \(J_m = \max(|J|)\) over the periodic box

black – spectral
red – GASpAR
light blue – FD

Note: the red curve nearly covers the black curve.
**Time series comparisons** \((\eta = \nu = 0.002, 128^2)\)

\[
\langle |\nabla J|^2 \rangle = \int_0^1 dx \int_0^1 dy |\nabla J|^2
\]

black – spectral  
red – GASpAR  
light blue – FD

Note: the red curve nearly covers the black curve.
Time series comparisons \((\eta = \nu = 0.002, 128^2)\)

\[
d(E_M + E_K)/dt \text{ by finite difference}
\]

black solid – spectral
red solid – GASpAR
light blue solid – FD

\[-\nu\langle \Omega^2 \rangle - \eta\langle J^2 \rangle\]
black dashed – spectral
green solid – GASpAR
blue solid – FD

Note: the green curve basically covers the red and black (solid and dashed) curves.
Time series comparisons \( (\eta = \nu = 0.002, 128^2) \)

\[
\left| \frac{d(E_M + E_K)}{dt} + \nu\langle \Omega^2 \rangle + \eta\langle J^2 \rangle \right| \left( \nu\langle \Omega^2 \rangle + \eta\langle J^2 \rangle \right)
\]

black solid – spectral
red solid – GASpAR
light blue solid – FD

\[
\left| \frac{dH_M}{dt} + \eta E_M \right| \eta E_M,
\]

with \( H_M = \frac{1}{2} \int_0^1 dx \int_0^1 dy |A|^2 \).

black solid – spectral
red solid – GASpAR
light blue solid – FD

(Note that \( A \) is a diagnostic that derives from a separate solve for the GASpAR code.)
**GASpAR grid** ($\eta = \nu = 0.001$, $256^2$)

$p = 8$ in GASpAR
d.o.f. = 43520

AMR finite difference code uses the same grid with each square having an $8\times8$ uniform grid within.

Pseudo spectral code uses a uniform grid.
d.o.f. = 131072
Time series comparisons \( (\eta = \nu = 0.001, 256^2) \)

Kinetic energy

\[
E_K = \frac{1}{2} \int_0^1 dx \int_0^1 dy |\nabla \phi|^2
\]

black dashed – spectral

green solid – GASpAR

blue solid – FD

Total energy: \( E_M + E_K \)

black solid – spectral

red solid – GASpAR

light blue solid – FD

where

\[
E_M = \frac{1}{2} \int_0^1 dx \int_0^1 dy |\nabla A|^2
\]

is the Magnetic energy.

Note that the light blue solid curve nearly covers the red and black solid curves; and the red dashed curve covers the black dashed curve.
Time series comparisons \((\eta = \nu = 0.001, 256^2)\)

\[
\langle J^2 \rangle = \int_0^1 dx \int_0^1 dy J^2
\]

black solid – spectral
red solid – GASpAR
light blue solid – FD

\[
\langle \Omega^2 \rangle = \int_0^1 dx \int_0^1 dy \Omega^2
\]

black dashed – spectral
green solid – GASpAR
blue solid – FD

Note: the red curve nearly covers the black solid, and the green covers the black dashed.
Time series comparisons \(( \eta = \nu = 0.001, 256^2 )\)

Maximum current density
\[ J_m = \max( |J| ) \]
over the periodic box

black – spectral
red – GASpAR
light blue – FD

Note: the red curve nearly covers the black curve.
Time series comparisons \( (\eta = \nu = 0.001, 256^2) \)

\[
\langle |\nabla J|^2 \rangle = \int_0^1 dx \int_0^1 dy |\nabla J|^2
\]

black – spectral
red – GASpAR
light blue – FD

Note: the red curve nearly covers the black curve.
**Time series comparisons** ($\eta = \nu = 0.001, 256^2$)

\[ \frac{d(E_M + E_K)}{dt} \] by finite difference
black solid – spectral
red solid – GASpAR
light blue solid – FD

\[-\nu \langle \Omega^2 \rangle - \eta \langle J^2 \rangle\]
black dashed – spectral
green solid – GASpAR
blue solid – FD

Note: the green curve basically covers the red and black (solid and dashed) curves.
**Time series comparisons** \((\eta = \nu = 0.001, 256^2)\)

\[
\left| \frac{d(E_M + E_K)}{dt} + \nu \langle \Omega^2 \rangle + \eta \langle J^2 \rangle \right| \\
\left( \nu \langle \Omega^2 \rangle + \eta \langle J^2 \rangle \right)
\]

black solid – spectral
red solid – GASpAR
light blue solid – FD

\[
\left| \frac{dH_M}{dt} + \eta E_M \right| \\
\eta E_M
\]

with \(H_M = \frac{1}{2} \int_0^1 dx \int_0^1 dy |A|^2\).
black solid – spectral
red solid – GASpAR
light blue solid – FD
**GASpAR grid** \( (\eta = \nu = 0.0003, \ 512^2) \)

\[ p = 8 \text{ in GASpAR} \]
\[ \text{d.o.f.} = 89600 \]

AMR finite difference code uses the same grid with each square having an \(8\times8\) uniform grid within.

Pseudo spectral code uses a uniform grid.
\[ \text{d.o.f.} = 524288 \]

\[ t = 0.93 \]
**Time series comparisons** \( (\eta = \nu = 0.0003, \ 512^2) \)

\[
\frac{d(E_M + E_K)}{dt} + \nu \langle \Omega^2 \rangle + \eta \langle J^2 \rangle \quad \text{and} \quad \left( \nu \langle \Omega^2 \rangle + \eta \langle J^2 \rangle \right)
\]

black solid – spectral  
red solid – GASpAR  
light blue solid – FD

\[
\left| \frac{dH_M / dt + \eta E_M}{\eta E_M} \right|
\]

with \( H_M = \frac{1}{2} \int_0^1 dx \int_0^1 dy |A|^2 \).

black solid – spectral  
red solid – GASpAR  
light blue solid – FD

(Note that \( A \) is a diagnostic that derives from a separate solve for the GASpAR code.)
Non-uniform adaptive grid scales almost linear with $N$, while a uniform grid scales with $N^2$. 
Cumulative accuracy comparison

Define

\[ d_i(t) = \int_0^t \left( \nu \langle \Omega^2 \rangle + \eta \langle J^2 \rangle \right) dt', \]

where \( i = \text{PS} \) for the pseudo-spectral run; \( i = \text{SE} \) for the spectral element GASpAR run; \( i = \text{FD} \) for the finite difference AMR run. Then define cumulative error function:

\[ \varepsilon_i(t) = \frac{|d_i(t) - d_{\text{PS}}(t)|}{|d_{\text{PS}}(t)|}. \]

128\(^2\) case

![Graph showing cumulative error functions \( \varepsilon_{\text{FD}}(t) \) and \( \varepsilon_{\text{SE}}(t) \).]
Cumulative accuracy comparison

256^2 case

512^2 case
Conclusion
◊ The spectral element MHD code GASpAR is applied for the problem of island coalescence instability in which strong current layers will form at known locations.
◊ Although GASpAR has the function of dynamic adaptive refinements, static non-uniform grids are used so as to compare results from a finite difference AMR code using the same grid. Results are also compared with a pseudo spectral code with uniform grids that has high accuracy.
◊ It is found that the degrees of freedom of the non-uniform grids that are sufficient to resolve current layers scale roughly with $N$, in contrast with the $N^2$ scaling of uniform grids.
◊ Various time series comparisons consistently show that the spectral element results agree better with the pseudo spectral results than those from the finite difference AMR runs, especially for quantities involving higher spatial derivatives.
◊ An analysis of cumulative errors also show that the results from GASpAR are of higher accuracy than those from the finite difference AMR code.
◊ Many aspects of the spectral element code still need to be developed, including extension to 3D.