THE ASTROPHYSICAL JOURNAL, 548:318–322, 2001 February 10

RANDOM SCATTERING AND ANISOTROPIC TURBULENCE OF SHEAR ALFVÉN WAVE PACKETS

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Received 2000 March 16; accepted 2000 October 11

ABSTRACT

A theoretical model is given of anisotropic magnetohydrodynamic turbulence in the interstellar medium and the solar wind. The model is motivated by observations that show significant deviations from the Kolmogorov power law. Dimensional and heuristic arguments are given and critically assessed. On the basis of the weak turbulence approximation in which three-wave interactions dominate, analytical and numerical results are obtained for the anisotropic energy spectrum produced by the random scattering of shear Alfvén waves propagating parallel to a large-scale magnetic field. The energy spectrum is shown to be proportional to $k_\perp^{-2}$, qualitatively consistent with some observations and wave kinetic theory.

Subject headings: magnetic fields — MHD — turbulence — waves

1. INTRODUCTION

Our observational knowledge of the small-scale density fluctuations in the ionized interstellar medium (ISM) is primarily due to interstellar scintillations (see the reviews by Rickett 1990 and Narayan 1992). These observations show two qualitatively important features of density fluctuation spectra: they obey power laws and are anisotropic. However, some delicate observational and theoretical issues complicate precise quantitative results on the power-law exponent(s) and the degree of anisotropy.

If we write the power spectrum in the form

$$P_N(\boldsymbol{k}) = C_N^2 k^{-\alpha}, \quad k_{\rm out} \le k \le k_{\rm in} ,$$

where $k = |\boldsymbol{k}|$ is the magnitude of the wavenumber and C_N^2 is a positive constant, a large number of observations report that the exponent α is approximately equal to 11/3 over many decades in k (Armstrong, Cordes, & Rickett 1981; Cordes, Weisberg, & Boriakoff 1985; Gwinn et al. 1988; Spangler & Cordes 1988; Mutel & Lestrade 1990). Equivalently, if the power spectrum is expressed as a one-dimensional form in k-space, the exponent is given by $\beta = \alpha - 2 \approx 5/3$ which is identical to that predicted by Kolmogorov's well-known inertial range spectrum for turbulent fluids (Kolmogorov 1941). Although this is suggestive (Lee & Jokipii 1975; Armstrong, Cordes, & Rickett 1981), it is far from obvious why Kolmogorov's spectral law for an incompressible and isotropic neutral fluid should apply at all to the ISM which is a compressible ionized medium permeated by a large-scale and directed magnetic field.

A definitive theoretical interpretation of the observed spectra is difficult for at least two reasons. First, the exponent α (or β) depends sensitively on the mechanism that produces the turbulence and needs to be determined with a high level of precision in order to discriminate between different theoretical models. A number of observations show that $\alpha(\beta)$ is less than 4(2), but this is not precise enough to discriminate between different theoretical models. Second, the claim that the power law should be attributed to an inertial range spectrum carries with it not only the challenge of establishing the power law over several decades in k, but also the identification of an outer wavenumber $k_{\rm out}$ where energy is injected and an inner wavenumber $k_{\rm in}$ above which energy is dissipated. The realization of both of these objectives simultaneously by independent measurements is difficult. There has been a general tendency among observers to settle for the Kolmogorov exponent $\beta = 5/3$ (Rickett & Lyne 1990; Spangler & Gwinn 1990; Gupta, Rickett, & Coles 1993; Wilkinson, Narayan, & Spencer 1994; Molnar et al. 1995). However, a significant number of observations show definite deviations from Kolmogorov scaling (Cordes & Wolszczan 1986; Moran et al. 1990; Wilkinson et al. 1994; Mutel, Molnar, & Spangler 1999, private communication; Lambert & Rickett 2000). We mention here a few examples. From observations of a scatter-broadened image of Cygnus X-3, Wilkinson et al. (1994) report exponents $\beta \approx 2.03 \pm 0.05$ and 1.93 ± 0.05 at frequencies of 408 and 1692 MHz, respectively. More recently, from observations of a scatter-broadened image of 2005 + 403 at 1.42 GHz, R. M. Mutel et al. (1999, private communication) report that $\beta \approx 1.93 \pm 0.03$. The error bars in these observations are sufficiently small that they raise serious questions regarding the universal validity of a Kolmogorov scaling for ISM turbulence. In fact, these examples suggest that β lies in the range between 1.9 and 2. Lambert & Rickett (2000) have shown recently that many features of diffractive measurements can be accounted for by a nonturbulent $\beta = 2$ model with abrupt (or discontinuous) changes in the density profile of the ISM (Blandford & Narayan 1985), but due to the presence of several discrepant features in the data they rule out both the $\beta = 2$ model as well as the Kolmogorov model as universal models for ISM fluctuations. They, therefore, suggest the development of different spectral models for different lines of sight.

As mentioned above, there is significant observational evidence to suggest that the interstellar scintillation spectrum is anisotropic. If the density irregularities were isotropic, the blurring pattern or "seeing disk" of a source viewed through the turbulence would be round and symmetric. It is actually observed that the blurring pattern is elongated (Spangler & Cordes 1988; Wilkinson et al. 1994; Frail et al. 1994; Molnar et al. 1995). The observed elongation of the scattering disk is attributed to anisotropy caused by the background interstellar magnetic field which causes the density irregularities to have a cigar-like structure, with long spatial scales parallel and short spatial scales perpendicular to the background field (Higdon 1984). The degree of anisotropy is different for different sources (that is, different lines of sight through the ISM). Averaging along the line of sight can cause a reduction in the measured degree of anisotropy.

Non-Kolmogorov power laws for velocity and magnetic field fluctuation spectra have also been observed in the turbulent solar wind. Observations from *Voyager 1* and *Voyager 2* spacecrafts between 13 and 25 AU show k^{-2} spectra at low frequencies at low heliographic latitudes (Burlaga & Goldstein 1984; Burlaga, Ness, & McDonald 1987). A possible explanation of this spectrum is that it is mainly due to the presence of shocks and discontinuities (Burlaga & Mish 1986; Roberts & Goldstein 1987). However, it is also possible that the spectra may have a turbulent origin with turbulence eventually steepening to produce shocks. As in the ISM, anisotropy is a persistent feature of solar wind turbulence and manifests itself in several in situ observations as more power perpendicular than parallel to the local magnetic field (Klein et al. 1993; Bieber, Wanner, & Matthaeus 1996).

In this paper, we present a new calculation of the anisotropic energy spectrum in a plasma permeated by a uniform background magnetic field. The closure employed in this calculation is weak turbulence (Sagdeev & Galeev 1969; Zakharov, Lvov, & Falkovich 1992) which has been shown to be dominated by three-wave interactions (Montgomery & Matthaeus 1995; Ng & Bhattacharjee 1996; Galtier et al. 2000). We show by means of a novel simulation of the random scattering of shear Alfvén waves that the inertial range anisotropic energy spectrum is proportional to $k_\perp^{-2}$, obtained earlier by heuristic analysis (Ng & Bhattacharjee 1997; Goldreich & Sridhar 1997) and wave kinetic theory (Galtier et al. 2000). Although the geometry of our model is simple and weak turbulence closure is restrictive, the calculation provides qualitative support for some of the observations on non-Kolmogorov power laws.

The following is an outline of this paper. In § 2, we review the dimensional and heuristic arguments for the Kolmogorov, Iroshnikov-Kraichnan (IK; Iroshnikov 1963; Kraichnan 1965), and anisotropic MHD energy spectra. We do so because although such arguments have been successful and are widely used, they can be problematic, reinforcing the need for careful dynamical calculations. In § 3, we present analytical and numerical calculations that test and verify the heuristic arguments for anisotropic weak MHD turbulence. We conclude in § 4 with a summary of our results, a discussion of the limitations of our theoretical model, and implications for observations of turbulence in the ISM and the solar wind.

2. DIMENSIONAL AND HEURISTIC ANALYSIS

Kolmogorov derived his celebrated energy spectrum for hydrodynamics (HD) essentially by dimensional analysis (Kolmogorov 1941). He made two crucial assumptions: the turbulence is isotropic and the dominant interactions between eddies are local in k-space. The latter implies that large eddies convect small eddies but do not affect the internal dynamics of the small eddies. In this picture, large-scale flows act like spatially uniform velocity fields that can be eliminated by a Galilean transformation without affecting small-scale dynamics.

If the turbulence is isotropic in k-space, the energy can be written $\int E(k)dk$, where $E(k)$ is the energy spectrum. Assume, following Kolmogorov, that there exists an inertial range such that the energy transfer rate $\varepsilon(k)$ is a constant independent of k and, furthermore, that the energy transfer process is local in k-space. Since the dimension of ε is L^2T^{-3}, k is L^{-1}, and $E(k)$ is L^3T^{-2} (where L is the dimension of length and T is of time), dimensional homogeneity of the relation $\varepsilon \sim k^\alpha E_k^\beta$ yields the inertial-range energy spectrum $E(k) \propto \varepsilon^{2/3}k^{-5/3}$.

Although Kolmogorov's spectrum has been verified by numerous laboratory and numerical experiments, vexing questions remain regarding the assumption of local interactions in k-space (see Chorin 1994, and references therein). The widely accepted picture of a local, self-similar cascade suggests that energy is transferred irreversibly from the large injection scale through the inertial range to the dissipation scale, with each shell in k-space transferring energy to a neighboring shell. If v_k^2 is the typical energy associated with a shell and $\tau_k \sim (kv_k)^{-1}$ is the characteristic time required to transfer energy from one shell to the next, the energy transfer rate is given by $\varepsilon \sim v_k^2/\tau_k \sim kv_k^3$. Hence, the energy associated with a shell can be written $v_k^2 \sim \varepsilon^{2/3}k^{-2/3}$. Chorin points out that in order to set up a self-similar cascade by means of local interactions in k-space, all of the energy in one shell must move to another in a characteristic time τ_k. This requires that the energy content as well as the energy transfer time be the same for every shell. However, inspection of the expression $v_k^2 \sim \varepsilon^{2/3}k^{-2/3}$ shows clearly that this requirement cannot be satisfied because $\varepsilon(k)$ is a constant independent of k and the $k^{-2/3}$ dependence implies that the energy of each shell actually decreases with increasing k. (The same is true of τ_k.) Chorin argues that for the cascade picture to hold, it is probably necessary to invoke non-local interactions, with energy sloshing back and forth between distant shells in k-space like water in a bathtub, and spilling over into the dissipation range to account for the overall loss of energy in a turbulent flow.

IK extend Kolmogorov's analysis to incompressible magnetohydrodynamic (MHD) turbulence. As discussed by Kraichnan (1965), the small wave number components act like a background magnetic field which cannot be removed by a Galilean transformation and support Alfvén wave packets propagating in both directions with the Alfvén speed V_A. An Alfvén wave packet can interact with another wave packet only if the two collide, with the interaction time given typically by $\tau_k \sim (kV_A)^{-1}$. Note that τ_k has an inherently nonlocal character in k-space because it depends not only on the typical spatial dimension (k^{-1}) of a wave (a local property) but also on the magnitude of the large-scale (small k) magnetic perturbation that determines V_A (a nonlocal property). (This type of nonlocal behavior in MHD has been known since the advent of the IK theory and is not the primary focus of our concern.) In many cases of physical interest $\tau_k \sim (kV_A)^{-1}$ is much shorter than the eddy turnover time $(kv_k)^{-1}$, with the consequence that the energy cascade is more inhibited in MHD than it is in HD. By treating the $k = 0$ component of the magnetic field at any spatial location as the background uniform field, and assuming that the energy cascade is isotropic and local in k-space, Kolmogorov's dimensional analysis arguments can be repeated, now with ε depending on k, $E(k)$ and V_A (with dimension LT^{-1}). Writing $\varepsilon \sim k^\alpha E_k^\beta V_A^\gamma$, we can deduce the spectral index ν of the inertial-range energy spectrum:

$$\nu = \frac{\alpha}{\beta} = \frac{5-\gamma}{3-\gamma} . \qquad (1)$$

(Note that the Kolmogorov spectrum $\nu = 5/3$ for HD is obtained as a special case of eq. [1] if we set $\gamma = 0$.) To find ν for MHD, we must determine γ. This can be done in the limit of weak turbulence when the lowest order interaction involves three-wave interactions during which two wave packets collide for a typical time scale $\tau_k \sim (kV_A)^{-1}$ and produce a third wave with typical velocity magnitude $\delta v_k \sim \dot{v}_k \tau \sim v_k^2/V_A$. By the relation $\varepsilon \propto (\delta v_k)^2/\tau_\kappa \propto V_A^{-1}$, we obtain $\gamma = -1$ and thus $\nu = 3/2$, which yields the IK spectrum $E(k) \propto \varepsilon^{1/2} k^{-3/2}$.

The scaling results obtained above by dimensional analysis for isotropic MHD turbulence can also be obtained by an alternate heuristic physical argument. Let each of the two colliding wave packets have amplitude of the order v_k and spatial scale k^{-1}. We assume again that the energy transfer is local in k-space, which means that a wave will interact dominantly with another wave characterized by the same length scale but moving in the opposite direction. From the MHD equations, we estimate that $\dot{v}_k \sim k v_k^2$, where an overdot denotes time derivative. If three-wave interactions dominate, we can write $\delta v_k \sim \dot{v}_k \tau \sim v_k^2/V_A$. In the weak limit, it will take a large number of random collisions, $N \sim (v_k/\delta v_k)^2 \gg 1$ to change a wave packet amplitude by a factor of unity. Noting that $E(k) \sim k^{-1} v_k^2$, we obtain $\varepsilon \sim v_k^2/N\tau \sim k^2 E(k) V_A/N \sim k^3 E^2(k) V_A^{-1}$ which implies again that $E(k) \propto \varepsilon^{1/2} k^{-3/2}$.

The IK theory, which has provided the physical underpinnings of much subsequent work on MHD turbulence, neglects anisotropy. Subsequently, numerous analytical and computational studies have attempted to address different aspects of anisotropic turbulence (Montgomery & Turner 1981; Shebalin, Matthaeus, & Montgomery 1983; Sridhar & Goldreich 1994; Oughton, Priest, & Matthaeus 1994; Goldreich & Sridhar 1995, 1997; Ng & Bhattacharjee 1996, 1997; Matthaeus et al. 1996; Chen & Kraichnan 1997; Galtier et al. 2000). In the presence of a uniform magnetic field, the spectrum is anisotropic. Dimensional analysis, by itself, cannot then provide a definite result since it cannot discriminate between the two length scales perpendicular ($k_\perp^{-1}$) and parallel ($k_\parallel^{-1}$) to the uniform magnetic field. Let us assume that the energy cascade occurs entirely in the direction perpendicular to the uniform field so that the total energy can be written $\int E(k_\perp) dk_\perp\, dk_\parallel$. (This assumption is shown to be true in § 3.) If we now repeat the scaling argument given in the last paragraph with $\tau_k \sim (k_\parallel V_A)^{-1}$ and $E(k_\perp) \sim (k_\parallel k_\perp)^{-1} v_k^2$, we obtain

$$\delta v_k \sim k_\perp v_k^2/(k_\parallel V_A) \sim k_\perp^2 E(k_\perp)/V_A . \quad (2)$$

Thus $\varepsilon \sim k_\perp^4 k_\parallel E^2(k_\perp)/V_A$ which implies that the anisotropic spectrum is $E(k_\perp) \propto \varepsilon^{1/2} k_\perp^{-2}$ for weak MHD turbulence dominated by three-wave interactions.

Chorin's reservations regarding Kolmogorov's dimensional analysis also apply to the heuristic arguments in support of the IK and anisotropic MHD spectra. In particular, for anisotropic MHD turbulence, we obtain $v_k^2 \sim \varepsilon^{1/2}(k_\parallel V_A)^{1/2} k_\perp^{-1}$ which shows that the energy transferred from one $k_\perp$-shell to the next is not constant. Here too the bathtub analogy suggests that energy sloshes back and forth nonlocally in $k_\perp$-space, with some of it spilling over into the dissipation range.

In a previous paper (Ng & Bhattacharjee 1997), we have reported the results of our effort to test the assumption that the energy transfer rate $\varepsilon(k)$ in MHD depends only on local interactions in k-space. One of the main results of our previous paper is that the heuristic scaling relation in equation (2) for three-wave interactions is not valid over a typical collision time between two oppositely propagating wave packets. Consequently, our earlier attempt to test the anisotropic spectrum $E(k_\perp) \propto \varepsilon^{1/2} k_\perp^{-2}$ was not conclusive. In § 3, we extend our previous calculation over many random collisions, and find rather remarkably that the anisotropic spectrum predicted by the heuristic argument emerges. This suggests that although energy transfer is not local in k-space for a typical collision between two oppositely propagating wave packets, when averaged over many random collisions the nonlocality of the energy transfer process does not appear to affect the scaling derived from the heuristic argument.

3. RANDOM THREE-WAVE INTERACTIONS AND THE ANISOTROPIC SPECTRUM

We assume for simplicity that the plasma fluid is permeated by a spatially uniform magnetic field $\boldsymbol{B} = \hat{z}$. Then the nonlinear MHD equations can be reduced rigorously to the so-called reduced MHD (RMHD; Strauss 1976; Zank & Matthaeus 1992; Bhattacharjee, Ng, & Spangler 1998) equations

$$\frac{\partial \Omega}{\partial t} - \frac{\partial J}{\partial z} = [A, J] - [\phi, \Omega] , \quad (3)$$

$$\frac{\partial A}{\partial t} - \frac{\partial \phi}{\partial z} = -[\phi, A] , \quad (4)$$

where the magnetic field is given by $\boldsymbol{B} = \hat{z} + \nabla_\perp A \times \hat{z}$ with A as the magnetic flux function, the flow velocity is given by $\boldsymbol{v} = \nabla_\perp \phi \times z$ with ϕ as the stream function, and $[\phi, A] \equiv \phi_y A_x - \phi_x A_y$. Here $\Omega = -\nabla_\perp^2 \phi$ is the parallel vorticity and $J = -\nabla_\perp^2 A$ is the parallel current density. Note that we have normalized the background uniform magnetic field in the z-direction to have unit magnitude, and the density has been chosen so that the Alfvén speed $V_A = 1$.

Recent work has demonstrated conclusively that weak MHD turbulence in the presence of a uniform magnetic field is dominated by three-wave interactions that mediate the collisions of shear-Alfvén wave packets (Montgomery & Matthaeus 1995; Ng & Bhattacharjee 1996, hereafter NB; Galtier et al. 2000). Using the ideal RMHD equations, NB calculate in closed form the three-wave and four-wave interaction terms, and show the former to be asymptotically dominant if the wave packets have nonzero $k_\parallel = 0$ components. These three-wave interaction terms provide the basis for our Monte Carlo simulation of the random scattering of Alfvén waves, discussed below.

For weak interactions between two colliding shear-Alfvén wave packets $f^\pm$ traveling in the $\pm z$ directions, we write perturbative solutions of the form

$$\phi = f^-(\boldsymbol{x}_\perp, z^-) + f^+(\boldsymbol{x}_\perp, z^+) + \phi_1 + \phi_2 + \dots , \quad (5)$$

$$A = f^-(\boldsymbol{x}_\perp, z^-) - f^+(\boldsymbol{x}_\perp, z^+) + A_1 + A_2 + \dots , \quad (6)$$

where $\boldsymbol{x}_\perp = (x, y)$ is perpendicular to z and $z^\pm = z \mp t$. Here $f^\pm(\boldsymbol{x}_\perp, z^\pm)$ represents Alfvén wave packets that propagate nondispersively with the Alfvén speed $V_A = 1$. For given zero-order fields $f^\pm$, we can then calculate the first-order fields from the equations

$$\frac{\partial \Omega_1}{\partial t} - \frac{\partial J_1}{\partial z} = 2\{[f^+, \nabla_\perp^2 f^-] + [f^-, \nabla_\perp^2 f^+]\} \equiv F , \quad (7)$$

$$\frac{\partial A_1}{\partial t} - \frac{\partial \phi_1}{\partial z} = 2[f^-, f^+] \equiv G . \quad (8)$$

Equations (7) and (8) are radiation equations for the first-order fields, with the source term determined by the overlap of the given zero-order fields f^+ and f^-. The asymptotic expressions of ϕ_1, A_1 can be written

$$\phi_1(\boldsymbol{x}_\perp, t \to \infty) \to f_1^-(\boldsymbol{x}_\perp, z^-) + f_1^+(\boldsymbol{x}_\perp, z^+) , \quad (9)$$

$$A_1(\boldsymbol{x}_\perp, t \to \infty) \to f_1^-(\boldsymbol{x}_\perp, z^-) - f_1^+(\boldsymbol{x}_\perp, z^+) , \quad (10)$$

where

$$f_1^\pm(\boldsymbol{x}_\perp, z) = \pi \int [\tilde{F}'(\boldsymbol{k}, \pm k_z) \mp \tilde{G}(\boldsymbol{k}, \pm k_z)] e^{i\boldsymbol{k}\cdot\boldsymbol{x}} d\boldsymbol{k} . \quad (11)$$

Here $\tilde{F}'(\boldsymbol{k}, \omega) \equiv \tilde{F}(\boldsymbol{k}, \omega)/k_\perp^2$ and $\tilde{F}(\boldsymbol{k}, \omega)$ is the Fourier transform of $F(\boldsymbol{x}, t)$, defined by

$$F(\boldsymbol{x}, t) = \int \tilde{F}(\boldsymbol{k}, \omega) e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} d\boldsymbol{k} d\omega .$$

The Fourier transform $\tilde{G}(\boldsymbol{k}, \omega)$ is similarly defined. For simplicity, we consider the case when the functions $f^\pm(\boldsymbol{x}_\perp, z)$ are separable, i.e., $f^\pm(\boldsymbol{x}_\perp, z) = f_\perp^\pm(\boldsymbol{x}_\perp) f^\pm(z)$. NB show that

$$\tilde{F}(\boldsymbol{k}, \omega) = \tfrac{1}{2} \tilde{F}_\perp(\boldsymbol{k}_\perp) \tilde{f}^+(\kappa^+) \tilde{f}^-(\kappa^-),$$

$$\tilde{G}(\boldsymbol{k}, \omega) = \tfrac{1}{2} \tilde{G}_\perp(\boldsymbol{k}_\perp) \tilde{f}^+(\kappa^+) \tilde{f}^-(\kappa^-) ,$$

where $\kappa^\pm \equiv (k_z \pm \omega)/2$, $\tilde{f}(\kappa^\pm)$ is the one-dimensional Fourier transforms of $f^\pm(z^\pm)$, and $\tilde{F}_\perp$ and $\tilde{G}_\perp$ are the two-dimensional Fourier transforms of $F_\perp$ and $G_\perp$. It follows that

$$f_1^\pm(\boldsymbol{x}_\perp, z^\pm) = \pi u_\perp^\pm(\boldsymbol{x}_\perp) \tilde{f}^\mp(0) f^\pm(z^\pm)/2 , \quad (12)$$

where

$$u_\perp^\pm(\boldsymbol{x}_\perp) = \int [\tilde{F}'_\perp \mp \tilde{G}_\perp] e^{i\boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp} d\boldsymbol{k}_\perp , \quad (13)$$

and $\tilde{f}^\pm(0)$ is the $k_z = 0$ Fourier component of $f^\pm(z)$, with $\tilde{F}'(\boldsymbol{k}_\perp) \equiv \tilde{F}(\boldsymbol{k}_\perp)/k_\perp^2$. We note that the expression (13) for three-wave interactions preserves the z-dependence of the zero-order fields. This implies that there is no energy transfer parallel to the magnetic field when three-wave interactions dominate (Shebalin et al. 1993; Oughton et al. 1994; NB; Goldreich & Sridhar 1997; Kinney & McWilliams 1998). [Note that this conclusion holds independent of the assumption of separability of the functions $f^\pm(\boldsymbol{x}_\perp, z)$.]

Imposing periodic boundary condition in $\boldsymbol{x}_\perp$, we write

$$f_\perp^\pm(\boldsymbol{x}_\perp) = \sum_{mn} f_{mn}^\pm e^{2\pi i(mx+ny)} , \quad (14)$$

where $f_{mn}^\pm$ are constants. We define the energy $\int E_\pm(k_\perp) dk_\perp$ with the spectral functions

$$E_\pm(k_\perp) \propto k_\perp^{-\mu_\pm} \text{ or } |f_{mn}^\pm| \propto (m^2 + n^2)^{-(3+\mu_\pm)/4} , \quad (15)$$

where $\mu_\pm$ are the spectral indices. Assuming that the energy is randomly distributed in the zeroth-order fields, we can calculate the spectra of the first-order fields using equation (12). Our main objective is to determine how the spectrum of an Alfvén wave packet changes in time after many collisions with wave packets coming from the opposite direction. To be specific, let us consider the evolution of a f^+ field interacting with a sequence of random f^- fields. We write

$$\frac{\partial \Psi^+}{\partial t} = -[f^-, \Psi^+] + [f_x^-, f_x^+] + [f_y^-, f_y^+] , \quad (16)$$

where $\Psi^+ = -\nabla_\perp^2 f^+$. Numerically, the Fourier amplitudes f_{mn}^- are randomly chosen for a given spectral index μ_- in every time step τ_A. The time step is chosen small enough so as to satisfy the weak turbulence assumption and to keep each wave packet in the sequence of f^- uncorrelated with any other in the sequence. Also, in order to realize a well-resolved inertial range for the f^+ spectrum, a hyperdissipation term of the form $\eta \nabla_\perp^6 \Psi^+$ is added to the right of equation (16). We optimize the simulation so that the inertial range index is insensitive to the value of η. Equation (16) is solved by a pseudospectral method for different values of μ_- and for different levels of resolution (up to 1024^2) until the f^+ spectrum reaches a quasi-steady state.

Figure 1 shows the f^+ spectra for the case with $\mu_- = 2$ for different levels of resolution. We see that the inertial range for all runs have roughly the same index, $\mu_+ \approx 2$. We have checked numerically the relation $\mu_+ + \mu_- \approx 4$ for a range of values of μ_+ and μ_-. For $\mu_+ = \mu_- = 2$, which corresponds to the case considered in § 2, we obtain the anisotropic energy spectrum $k_\perp^{-2}$, consistent with the heuristic result.

Before we conclude this section, we remark that Goldreich & Sridhar (1997) have obtained the same anisotropic energy spectrum for what they describe as "intermediate turbulence." They agree with NB that three-wave interactions dominate individual collisions between shear-Alfvén wave packets, but argue that interactions of all orders make comparable contributions to the so-called intermediate turbulent energy cascade. They also argue that perturbation theory is simply not applicable to intermediate turbulence. Although perturbation theory associated with weak turbulence has a limited domain of applicability and eventually does break down as the turbulence grows stronger, it is interesting that the calculation presented in this

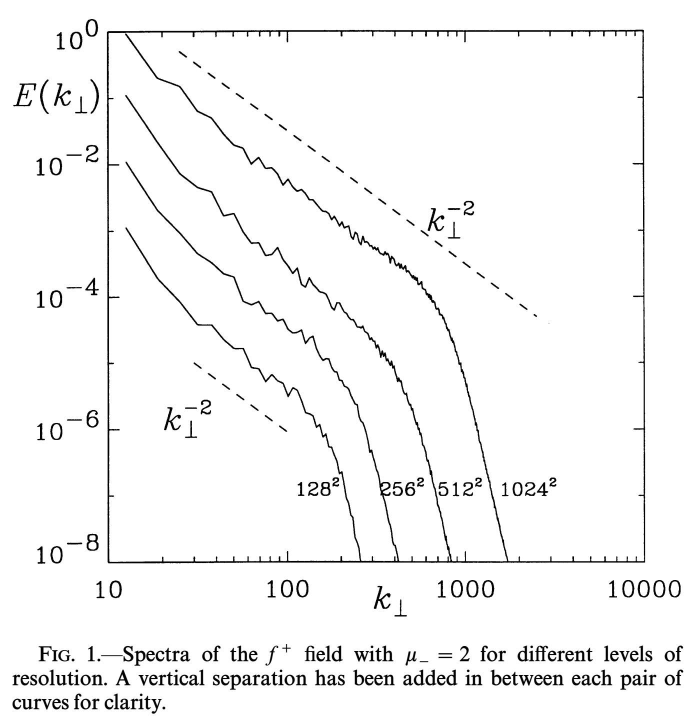

FIG. 1.—Spectra of the f^+ field with $\mu_- = 2$ for different levels of resolution. A vertical separation has been added in between each pair of curves for clarity.

paper actually yields the same spectrum as Goldreich & Sridhar's heuristic spectrum for intermediate turbulence, notwithstanding their reservations on the validity of perturbation theory. [See Nazarenko, Newell, & Galtier (2001) for a more detailed discussion of the validity of weak turbulence theory and the argument of Goldreich & Sridhar 1997.]

4. CONCLUSION

Prompted by observations of non-Kolmogorov and anisotropic turbulent spectra in the ISM and the solar wind, we have discussed a theoretical model of weak MHD turbulence, which produces an anisotropic energy spectrum proportional to $k_\perp^{-2}$. The anisotropic energy cascade in our model is due to the random scattering of shear-Alfvén waves dominated by three-wave interactions. We have reviewed critically the assumptions underlying the heuristic derivation of scaling laws in HD and MHD turbulence and underscored the need to verify these scaling laws by dynamical calculations. Our dynamical calculation provides independent confirmation of the $k_\perp^{-2}$-spectrum derived earlier from heuristic arguments (Ng & Bhattacharjee 1997; Goldreich & Sridhar 1997) and wave kinetic theory (Galtier et al. 2000).

Since measurements of fluctuations in the ISM are line-integrated, an interesting question is how the anisotropic spectrum obtained above for a uniform magnetic field might show up in observations of the ISM, which is generally permeated by a spatially varying magnetic field. If we make the drastic but simplifying assumption that the background magnetic field $\boldsymbol{B}_0$ takes all possible directions with equal probability, it is easy to show by averaging over three-dimensional wave vector space that the spectrum will be proportional to k^{-2} (Galtier et al. 2000). However, because all possible directions for $\boldsymbol{B}_0$ are not equally probable, one might expect deviations from the k^{-2} scaling. This remark is also applicable to observations of anisotropic turbulence in the solar wind which are based so far entirely on single spacecraft observations and are thus unable to provide quantitative information on the precise three-dimensional structure of anisotropic MHD turbulence (see Horbury 2000 for a recent review).

We conclude with a few cautionary remarks on the limitations of our model. Although one of the strengths of weak turbulence theory is that it provides rigorous closure, it is far from clear that weak turbulence is a valid approximation for ISM turbulence, which is often strong and compressive. Furthermore, we have calculated energy spectra, not density fluctuation spectra. It is often assumed that density fluctuations are enslaved to energy fluctuations, but this is not necessarily so (Bhattacharjee et al. 1998; Terry, Fernandez, & Ware 1998). In future work, we will attempt to remedy some of these limitations.

We thank R. Bruno, S. Galtier, R. Mutel, S. Nazarenko, A. Newell, A. Pouquet, and S. Spangler for useful discussions. This research is supported by the NSF Grant ATM-0001317. Numerical computations were carried out on the National Partnership for Advanced Computational Infrastructure.

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