Anisotropic MHD Turbulence

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6th Annual International Astrophysics Conference,
Honolulu, Hawaii, 20 March 2007

Support: NSF
Outline

• 2D MHD turbulence simulations -- globally isotropic with local anisotropy
• Global anisotropy due to magnetic field
• Weak MHD turbulence
• Comparisons with recent observations
• Conclusion
Hydrodynamic turbulence

Kolmogorov (1941): can get energy spectrum by dimensional analysis

Assumptions: ♦ isotropy
♦ local interaction in $k$-space
(energy moves from one $k$-shell to the next)

Energy cascade rate: $\varepsilon(k) \propto k^\alpha E(k)^\beta = \text{constant}$

$\Rightarrow \alpha = 5/2, \quad \beta = 3/2$

Kolmogorov spectrum: $E(k) \sim C_K \varepsilon^{2/3} k^{-5/3}$

Kolmogorov constant: $C_K \sim 1.4 - 2$
Interstellar turbulence

Observation: power law relation between electron density spectrum and spatial scales

From Cordes (1999)
Solar wind turbulence

Observation: power law in magnetic energy spectrum

From Goldstein & Roberts (1999)
MHD turbulence

Additional dependence on the Alfvén speed $V_A$

Energy cascade rate: $\varepsilon(k) \propto k^\alpha E(k)^\beta V_A^\gamma = \text{constant}$

$\Rightarrow \alpha = (5 - \gamma)/2, \quad \beta = (3 - \gamma)/2$

Spectral index: $\nu = \alpha / \beta = (5 - \gamma)/(3 - \gamma)$

$\gamma = 0$: Kolmogorov spectrum $E(k) \sim C_K \varepsilon^{2/3} k^{-5/3}$

$\gamma = -1$: IK spectrum $E(k) \sim C_{IK} \varepsilon^{1/2} V_A^{1/2} k^{-3/2}$

Iroshnikov (1963), Kraichnan (1965) $C_{IK} \sim 1.8 - 2.2$

Dimensionless parameter: $\chi \equiv k^{1/2} E_k^{1/2} V_A^{-1} = \nu_k / V_A \ll 1$
Weak MHD turbulence

Alfvén effect: cascade develops only if two Alfvén wave packets propagating in opposite directions collide

- Weak turbulence: $\chi << 1$
- Time scales:
  - Eddy turn-over time $\tau_N \sim 1/k\nu_k$
  - Alfvén time $\tau_A \sim 1/kV_A = \chi\tau_N << \tau_N$
  - Energy cascade time $\tau_E \sim \tau_A / \chi^2 = \tau_N / \chi >> \tau_N$

Kolmogorov cascade rate: $\varepsilon_K \sim v_k^2 / \tau_N \sim k^{5/2}E_k^{3/2}$

IK cascade rate: $\varepsilon_{IK} \sim v_k^2 / \tau_E \sim \varepsilon_K \chi \sim k^3E_k^2V_A^{-1} << \varepsilon_K$
Globally isotropic

• simulations of 2D MHD turbulence
• no $\mathbf{B}_0$
• decaying
• periodic in $x,y$
• $2048^2$

2D energy spectrum (logarithmic contours)
Local anisotropy

♦ simulations of 2D MHD turbulence
♦ no $B_0$
♦ decaying
♦ periodic in $x,y$
♦ $2048^2$
Current sheet formation

- intense current “sheets”
- along local $\mathbf{B}$
- local anisotropy

![current density $J$]
$k^{-3/2}$ energy spectra

- overlapping spectra over a period of large scale Alfvén time
$k^{-3/2}$ energy spectrum

- consistent with IK spectrum
- kinetic spectrum less steep than magnetic spectrum
- kinetic energy less than magnetic energy
Kinetic vs magnetic spectrum -- observations

From [Podesta, Roberts, & Goldstein 2006], power spectra for the total kinetic energy (a) and for the total magnetic energy (b) The best fit straight line over the interval is indicated by the short red line segment.
From [Podesta, Roberts, & Goldstein 2006], Alfvén ratio, the ratio of kinetic to magnetic energy spectra in the previous figure. The sharp rise in the Alfvén ratio beyond approximately $2 \times 10^{-2}$ Hz is not a real effect but is caused by aliasing in the velocity spectrum.
Energy cascade rate

Fit numerical values of

\[ E(k) = C_K \varepsilon^{2/3} k^{-5/3} = C_{IK} \varepsilon^{1/2} V_A^{1/2} k^{-3/2} \]

or \( \varepsilon = C_K^{-3/2} \varepsilon_K = C_{IK}^{-2} \varepsilon_{IK} \)

where

\[ \varepsilon_{IK} = k^3 E_k^2 V_A^{-1} \]

by IK theory

\[ \varepsilon_K = k^{5/2} E_k^{3/2} \]

by Kolmogorov theory

More consistent with IK theory

c.f. usually found values: \( C_K \sim 1.4 - 2, C_{IK} \sim 1.8 - 2.2 \)
Energy cascade rate

\[ \varepsilon \]

Amplitude changing rate

\[ \varepsilon_A \]

Energy cascade rate

\[ \varepsilon_E = k^3 E_k^2 V_A^{-1} \]

\[ \varepsilon_K = k^{5/2} E_k^{3/2} \]

More consistent with IK theory
Energy cascade rate -- observations

From [Vasquez et al. 2007, preprint], scatter plot of the Kolmogorov cascade rate (red diamonds - left), the IK cascade rate (red diamonds -right), and expected heating rate (blue crosses) as a function of proton temperature $T_{pr}$. The Kolmogorov rate exceeds the expected one by a factor of 10 or more, while the IK rate is roughly in agreement.
Anisotropy due to magnetic field

- Energy spectrum $E(k_{\parallel},k_{\perp})$ relative to a background $B_0$
- Dimension analysis cannot fix the form of $E(k_{\parallel},k_{\perp})$, although 1D integrated spectrum $E(k)$ remains the same form
- Energy cascade faster in $k_{\perp}$ than $k_{\parallel}$
- Alfvén time $\tau_A \sim 1/k_{\parallel}V_A$, $\chi \equiv k_{\perp}v_k / k_{\parallel}V_A$
- Critical balance cascade $\chi \sim 1$ (Goldreich & Sridhar 1995)
  recover Kolmogorov spectrum: $E(k_{\perp}) \sim \varepsilon^{2/3}k_{\perp}^{-5/3}$
  scale-dependent anisotropy: $k_{\parallel} \sim k_{\perp}^{2/3}L^{-1/3}$
Weak turbulence

Keep only three-wave interaction.
Possible if $\chi \sim k_\perp v_k / k_\parallel V_A << 1$

◆ Anisotropic energy spectrum: energy $= \int E(k_\parallel, k_\perp) dk_\parallel dk_\perp$

◆ No parallel energy cascade ($k_\parallel$ as a parameter)

◆ Energy cascade rate: $\varepsilon \sim \nu_k^2 / \tau_E \sim \nu_k^2 \tau_A / \tau_N \sim k_\perp^4 k_\parallel E^2(k_\perp) / V_A$

◆ Energy spectrum: $E(k_\perp) \propto \varepsilon^{1/2} k_\perp^{-2}$
Random collision

Energy spectrum settles after a series of random collision

Keep only three-wave interactions based on analytic expression.

\[ E(k_{\perp}) \propto k_{\perp}^{-2} \] found.

Same results from kinetic equation that obtains 2D spectrum of the form

\[ E(k_{\parallel}, k_{\perp}) \propto k_{\perp}^{-2} f(k_{||}) \]

(Galtier et al. 2000, 2005)

(Bhattacharjee & Ng 2001)
2D energy spectrum in simulations

- simulations of 2D MHD turbulence
- $B_0 = 2 \hat{y}$
- decaying
- periodic in $x,y$
- $2048^2$

2D energy spectrum $E(k_\parallel, k_\perp)$

(logarithmic contours)
2D energy spectrum in simulations

Note: for $B_0 = 0$, $E_{2D}(k, \theta = \text{const.}) \propto k^{-5/2}$ if $E_{1D}(k) \propto k^{-3/2}$
for $B_0 \neq 0$, still $E_{2D}(k, \theta \neq 0 \text{ or } \pi/2) \propto k^{-5/2}$ for $E(k_{\parallel}, k_{\perp}) \propto k_{\perp}^{-2} k_{\parallel}^{-1/2}$

$k^{3/2} E_{1D}(k)$

$k_{\perp}^{3/2} E_{2D}(k_{\parallel} = 0, k_{\perp})$

$k^2 E_{1D}(k)$

$k_{\perp}^2 E_{2D}(k_{\parallel} = 0, k_{\perp})$
Observations of $k^{-2}$ spectra

Interstellar media observation of the scatter-broadened image of Cygnus X-3

From Wilkinson et al. (1994)
Observations of $k^{-2}$ spectra

ISM observation of the scatter-broadened image of 2005+403

From Mutel, Molnar & Spangler (1999)
Observations of $k^{-2}$ spectra

In situ solar wind observation

From Burlaga & Goldstein (1984); Burlaga & Mish (1986); Roberts & Goldstein (1987)
Scale-dependent alignment between $\mathbf{v}$ and $\mathbf{B}$

$$v_k = v(x + r) - v(x) \quad B_k = B(x + r) - B(x)$$

$$\theta_k \equiv \langle \text{angle between } v_k \text{ and } B_k \rangle$$

Theoretical prediction [Boldyrev, 2005, 2006] for 3D driven MHD turbulence: $\theta_k \propto r^{1/4}$ if $r \perp B_0$
Scale-dependent alignment between $\mathbf{v}$ and $\mathbf{B}$

Observations in solar wind [Podesta et al., 2007, preprint]:

Comparison between the 1/4 scale dependence predicted by Boldyrev’s theory (dashed line) and the results from high-speed wind (magenta and red lines, respectively) and the first and second intervals of low-speed wind (blue and green lines, respectively).
Scale-dependent alignment between $v$ and $B$

2D simulations with $B_0$

- shows some scale-dependence close to $r^{1/4}$

- Boldyrev’s theory not applicable for this case

- still need to find the reason for such scale-dependence in this 2D case
Conclusion

• Globally isotropic 2D MHD turbulence is found to be weak with IK spectrum $E(k) \propto k^{-3/2}$ and energy cascade rate despise local anisotropy.

• Weak MHD turbulence with a strong magnetic field has anisotropic spectrum $E(k_\perp) \propto k_\perp^{-2}$ from random wave collision, kinetic equation, and direct simulation.

• Studies of 2D/3D MHD turbulence may help understanding of recent observations about the differences in magnetic and kinetic spectra, energy cascade rate, and scale dependence of the alignment of velocity and magnetic fluctuations.