Relation Between Emitted and Received Powers from a Moving Radiating Source in a Medium

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It is proposed that the ratio of the emitted power to the received power from a moving radiating source in a medium is to be considered and evaluated at constant $\omega_0$, the proper frequency characteristic of the source, instead of at constant $\omega$, the frequency of the radiation received. The general formula thus obtained differs from that given by Ko and Chuang (1973, Astrophys. Letters, 15, 125), even in the case of an isotropic medium. Application to a field-free plasma is made, which includes the situation of the complex Doppler effect, and the implication of the results to the case of a magnetoplasma is discussed.

There are two ways to calculate the energy flux from a radiating source. One is to find the far field (i.e., the field in the far zone) and in terms of which the energy flux is obtained; the other is to find the (net) work done by the source on the wave field of a certain wave-vector, to interpret it as the energy carried away by the outgoing wave, and the energy flux is thus also obtained. Either way can be used to give an expression for the radiation power per unit solid angle. In the case of a moving source (e.g., in synchrotron radiation where electrons move in helices), the two results differ from each other. This difference has caused some confusion until Ginzburg et al. (1968) and Scheuer (1968) correctly interpret that the power per unit solid angle calculated from the far-field method (hereafter denoted as $dP'/d\Omega$) is that received by a stationary observer and the same quantity calculated from the work-done method (hereafter denoted as $dP'/d\Omega$) is the power emitted by the source, both in the frame of the stationary observer. They also give the correct formula for the ratio $dP'/dP$ in the case of ultra-relativistic synchrotron radiation in vacuum. In fact, such a ratio has earlier been given by Jackson (1962).

Study of the relation $dP'/dP$ has been extended to a general medium. A number of years ago in this Letters, Ko and Chuang (1973) examined the then existing formulae for this relation in case of synchrotron radiation from a gyrating electron in a homogeneous magnetoplasma and obtained a compact formula. The absolute value of the following ratio

$$ (dP'/dP)_{KC} = 1 - V[\partial (k \cos \theta)/\partial \omega], $$

(1)

is their result, where $V$ is the velocity of the electron parallel to the magnetic field, $\theta$ and $\theta_0$, are respectively the angle of the wave-vector $k$ and the angle of the group velocity (i.e., the velocity of energy propagation) made with the field line, and the partial derivative with respect to the wave-frequency $\omega$ is taken at
constant $\theta_r$. (Note that Ko-Chuang's notations are slightly different from ours: their $v \cos \psi$, $P$, and $\bar{P}$ are simply our $V$, $dP'$, and $dP$, respectively.) An essential step in reaching this simple formula is the assumption of a constant $\omega$ in the evaluation of the change of the group velocity with angle. Our purpose here is to point out that $\omega_0$, the proper frequency characteristic of the source (which may refer to the frequency in one harmonic in the case of synchrotron radiation), rather than $\omega$, should be kept constant, and that the new result could differ from (1) substantially and qualitatively in several cases of interest.

To explain our viewpoint in more detail, we reconsider the problem as follows. Consider a certain amount of energy emitted by a moving source within an infinitesimal solid angle $d\Omega$ in a time interval $\Delta t'$ and that very amount of energy received by a distant observer in a corresponding time interval $\Delta t$. Because $(dP'/d\Omega) \Delta t' = (dP/d\Omega) |\Delta t|$ for small $\Delta t'$, the derivation of $dP'/dP$ is thus reduced to the problem of finding the ratio of the two time intervals. Now, the time of emission $t'$ and the time of reception $t$ are related by $t = t' + R/V_g$, where $R$ is the distance from the source at $t'$ to the observer, and $V_g$ is the speed of energy propagation. We have, therefore,

$$\Delta t = \Delta t' + \Delta R/V_g - R \Delta V_g/V_g^2$$

in a straightforward way. While it is easy to obtain $\Delta R = -V \cos \theta_r \Delta t'$ by simply projecting the distance travelled by the source onto the direction of energy propagation, it is a bit subtle in the evaluation of $\Delta V_g$. No doubt the change of $V_g$ comes from the change of $\theta_r$, i.e.,

$$\Delta V_g = (\partial V_g/\partial \theta_r) \Delta \theta_r = (\partial V_g/\partial \theta_r) V \Delta t' \sin \theta_r/R,$$

but how to carry out the partial derivative in (3) makes the difference. Ko and Chuang did it by taking $\omega$, the frequency of the received radiation, to be constant and eventually reached at (1). However, because of the Doppler effect, the wave frequency generally changes with angle according to

$$\omega = \omega_0/\gamma + \beta c k \cos \theta$$

where $\omega_0$, referred to as the proper frequency, is the frequency of the radiation measured in the frame co-moving with the source and $\gamma = 1/(1 - \beta^2)^{1/2}$, with $\beta = V/c$, is the usual relativistic factor. This means that $\omega$ should not be kept constant. Instead, because two different time intervals of emission and reception of the same amount of energy are being compared, the proper frequency $\omega_0$, characteristic of the source, should be kept constant in the evaluation of the partial derivative.

The subsequent derivation parallels very much that of Ko and Chuang. If the dispersion relation of the wave under consideration is given by $k = k(\omega, \theta)$, the two angles $\theta$ and $\theta_r$ are related by (Stix 1962)

$$\tan (\theta - \theta_r) = (\partial k/\partial \theta)_{\omega_0}/k$$

and the speed of energy propagation is (Ginzburg 1964)

$$V_g = [(\partial k/\partial \omega)_{\theta} \cos (\theta - \theta_r)]^{-1}.$$
The six quantities $\omega$, $k$, $\theta$, $\theta_r$, $V_g$, and $\omega_0$ in the four equations allow two independent variables. Transformation from $(\omega, \theta_r)$ to $(\omega_0, \theta_r)$ leads to

$$
(\partial V_g / \partial \theta_r)_{\omega_0} = (\partial V_g / \partial \omega)_{\theta_r}(\partial \omega / \partial \theta_r)_{\omega_0} + (\partial V_g / \partial \theta_r)_{\omega_0}
$$

and we see that the last term is just the one used by Ko and Chuang. Accepting the whole (7) as the correct expression for the partial derivative in (3) and following the same manipulation as given by Ko and Chuang, we readily obtain

$$
dP'/dP = |(dP'/dP)_{KC} + \Delta(dP'/dP)|
$$

where the first term on the right is given by (1), and

$$
\Delta(dP'/dP) = -V \sin \theta_r(\partial V_g / \partial \omega)_{\theta_r}(\partial \omega / \partial \theta_r)_{\omega_0} / V_g^2
$$

The additional term in (9) stems from (i) the Doppler effect where $\omega$ changes with angle at fixed $\omega_0$ and (ii) the dispersive effect where $V_g$ is a function of $\omega$ at fixed $\theta_r$; it could be non-zero even in the absence of anisotropy. Note that the foregoing consideration applies to any radiating source, not necessarily restricted to a gyrating electron, so long as the motion of the source is parallel to the magnetic field. For a source moving obliquely, there are additional angular variables to be considered and the result is expected to be much more complicated.

Applying to a radiating source of proper frequency $\omega_0$ moving in an isotropic plasma where the dispersion relation is given by $ck = (\omega^2 - \omega_p^2)^{1/2}$, we have

$$
(dP'/dP)_{KC} = 1 - \beta \cos \theta / N
$$

and

$$
\Delta(dP'/dP) = -\beta^2 \omega_0^2 N \sin^2 \theta / [c^2 \beta^2 (\beta \cos \theta - N)]
$$

where

$$
N = ck / \omega = ck / (\omega_0 / \gamma + \beta ck \cos \theta)
$$

is the refractive index, and

$$
ck = [\beta \gamma^{-1} \omega_0 \cos \theta \pm (\beta^2 \omega_0^2 \cos^2 \theta + \omega_0^2 / \gamma^2 - \omega_p^2)^{1/2}] / (1 - \beta^2 \cos^2 \theta)
$$

gives the wave number of the radiation received by the observer as a function of the angle of emission. Note that $\omega_0$ has been assumed larger than the plasma frequency $\omega_p$ in order to have radiation. Before going any further, it is of interest to notice the appearance of two solutions of $k$ for a fixed $\theta$ in (13). Bearing in mind that only positive $k$'s are acceptable, it is not difficult to see that, for a slow-moving source such that $\omega_0 / \gamma > \omega_p$, only the plus-signed solution is physical whereas, for a fast-moving source such that $\omega_0 / \gamma < \omega_p$, both solutions are valid for $\theta$ smaller than $\theta_{\text{max}} = \cos^{-1}[(\omega_p^2 - \omega_0^2 / \gamma^2)^{1/2} / \beta \omega_p]$. The existence of two $k$'s or two $\omega$'s of the same wave mode along one angular direction, known as the complex Doppler effect, was first discussed by Frank (1943) and later by others [Barsukov (1959), Papas (1965), and Berger (1976)]. It is also not difficult to realize that, in such a situation, all radiation will be confined within the forward cone of apex angle $2\theta_{\text{max}}$. Another illustrative way to appreciate this effect is to
recognize that (13) is nothing but the surface of intersection between the $\omega$ vs $ck$ dispersion surface (which is $\omega^2 = \omega_p^2 + c^2k^2$ for an isotropic plasma) and the $\omega$ vs $ck_{\parallel}$ ($k_{\parallel} = k \cos \theta$) plane defined by (4). This surface may be referred to as the Doppler-shifted wave-vector surface or DWS in brief and it is a prolate spheroid in the present case. Because (i) the plane is generally of slope $\beta$ and of $\omega$-intercept $\omega_0/\gamma$ and (ii) the dispersion surface has a cutoff frequency $\omega_p$, the plane gets more tilted and lowered for a larger $V$ and eventually cuts the dispersion surface to form a DWS only in the positive $k_{\parallel}$-space when $\omega_0/\gamma < \omega_p$; this gives only forward waves and the existence of two $k$'s for a fixed $\theta < \theta_{\text{max}}$.

We now compare the new result according to (8) with that of Ko and Chuang. For the normal case where $\omega_0/\gamma > \omega_p$, the two results are obviously the same at $\theta = 0^\circ$ and $180^\circ$, but at $\theta = 90^\circ$, they differ by the following fraction

$$\Delta(dP'/dP)_{\text{Ko}} = \beta^2 \omega_0^2 / (\omega_0^2 / \gamma^2 - \omega_p^2)$$

(14) 

$(dP'/dP)_{\text{KC}}$ being equal to unity. Therefore, a substantial difference appears around $\theta = 90^\circ$ provided that the motion of the source is relativistic, or the proper frequency $\omega_0$ is close to the cutoff frequency $\omega_p$, or both. For the complex Doppler case where $\omega_0/\gamma < \omega_p$, we may examine the two results near the maximum angle where the two solutions in (13) merge into one and $N$ becomes equal to $\beta \cos \theta_{\text{max}}$. We have $|(dP'/dP)_{\text{KC}}| \sim 0$ or, the received power per emitted power $|(dP'/dP)_{\text{KC}}| \sim \infty$ while, according to (11), $dP'/dP \sim \infty$ or

$$dP/dP' \sim 0$$

(15)

around $\theta_{\text{max}}$; they are two completely different results! One might think that, as $N$ approaching $\beta \cos \theta$, the speed of energy propagation $V_g (= cN)$ tends to $V \cos \theta$ so that all the wave energy emitted in $\Delta t'$ are squeezed into a very small $\Delta t$ upon reception and thus Ko-Chuang’s result is acceptable. However, such a consideration ignores the change of $V_g$ with $\theta$ according to (7); in fact, $(\partial V_g / \partial \theta)_{\text{app}}$ is large for $\theta \sim \theta_{\text{max}}$ and this means that the waves emitted in a short interval $\Delta t'$ have very different group velocities along such a direction, hence the observer can receive all of them only in a long interval $\Delta t$. So in our judgement, (15) gives the correct answer. This conclusion is also consistent with the earlier work by Lai and Chan (1986) where the received power and the emitted power were calculated separately through the concept of the Doppler-shifted wave-vector surface and compared.

The power ratio as a function of $\theta$ can be computed from (8) by substituting (12) and (13) into (10) and (11). A set of angular patterns of $dP/dP'$ (i.e., received power/ emitted power given by the radial length), rather than $dP'/dP$, are given by the solid curves in Figure 1 for $\omega_0/\omega_p = 1.1$ and at $\beta = 0$, 0.2, 0.4 and 0.5; the $\beta = 0$ curve is obviously circular with unit radius and may serve as a reference. The dashed lines give the corresponding $|(dP'/dP)_{\text{KC}}|$'s. Because the critical velocity for the complex Doppler effect is 0.4166 for $\omega_0/\omega_p = 1.1$, there are two values of $dP/dP'$, shown by the two solid curves, for each $\theta < \theta_{\text{max}} = 52.5^\circ$ at $\beta = 0.5$; similarly, there are two values from Ko-Chuang’s formula, shown by the two dashed curves. Note that, as $\theta$ approaching $\theta_{\text{max}}$, $dP/dP'$ tends to zero while $|(dP'/dP)_{\text{KC}}|$ tends to infinity as expected from the earlier
discussion. Incidentally, it is of interest to point out that $\Delta t/\Delta t'$ is negative for the lower-value curve at $\beta = 0.5$, meaning that the energy emitted earlier arrives at the observation point at a later time.

The foregoing example, amenable to analytic study, concerns a moving source in a field-free plasma. What would be the case if the synchrotron radiation in a magnetoplasma is considered? In such a gyrotrropic medium, there are in general five distinct sheets of dispersion surface (Oakes et al. 1979) instead of one, and the DWS corresponding to each of them, except in certain limits, cannot be expressed in a simple form like (13). Three of these sheets, with different cutoff frequencies, are nevertheless topologically the same as the simple one in the field-free plasma case and we therefore anticipate that qualitatively similar results will be obtained from our general formulae (8) and (9) for the synchrotron radiation in a magnetoplasma. We hope to come back to this problem in a later communication.

REFERENCES
Lai, H. M., and Chan, P. K., 1986, Phys. Fluids 29, 1881. (ω₀ in this paper should be replaced by ω₀/γ if the relativistic effect is non-negligible).