

Weakly Collisional Landau Damping and 3D BGK Modes: New Results on Old Problems

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Background

High temperature plasma is highly collisionless.

Two classic results in a collisionless plasma:

Landau damping of linear electrostatic waves

Landau (1946)

Exact nonlinear 1D electrostatic solutions (BGK modes)

---- (undamped) *Bernstein, Greene & Kruskal (1957)*

Two new results on these two classic problems.

Vlasov-Poisson equations

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} - \frac{q_s}{m_s} \nabla \psi \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0$$

$$\mathbf{E} = -\nabla \psi$$

$$\nabla \cdot \mathbf{E} = -\nabla^2 \psi = 4\pi \rho = 4\pi \sum_s q_s \int d\mathbf{v} f_s$$

Vlasov-Poisson equations -- linear solution

$$f = f_0 + f_1$$

$$\nabla^2 \psi = 4\pi \int d\mathbf{v} f_1$$

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{r}} + \frac{e}{m_e} \nabla \psi \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0$$

Plasma oscillation: *Tonks & Langmuir (1929), Vlasov (1938), Bohm & Gross (1949)*

Collisionless damping: *Landau (1946)*

Question: *why are plasma waves damped even without collision?*

Case-Van Kampen modes [Van Kampen (1955), Case (1959)]

Vlasov-Poisson equation for a single Fourier mode:

$$f_1(z, t, v) = e^{i(kz - \omega t)} \tilde{f}_1(k, \omega, v)$$

$$g = v_0 \sqrt{2} f_1 / n_0$$

$$(u - \Omega)g(u) = \eta(u) \int_{-\infty}^{\infty} g(u') du'$$

exact solution (singular, undamped like BGK modes):

$$g_{\Omega}(u) = P \left[\frac{\eta(u)}{u - \Omega} \right] + \delta(u - \Omega) \left[1 - P \int_{-\infty}^{\infty} \frac{\eta(u')}{u' - \Omega} du' \right]$$

Eigenmodes are the Case-Van Kampen modes.
Landau-damped modes are *not* eigenmodes, but long-time remnants of an arbitrary smooth initial condition.

Case-Van Kampen modes form a complete set

A general solution :
$$g(z,t,u) = e^{iz} \int_{-\infty}^{\infty} e^{-i\Omega t} c(\Omega) g_{\Omega}(u) d\Omega$$

Phase-mixing of undamped Case-Van Kampen modes produces Landau damping of the plasma wave.

The linear Vlasov-Poisson is completely solved.

Question: *what is the effect of collision, even if it is weak?*

A model equation : *Lenard & Bernstein (1958)*

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial z} - \frac{eE}{m} \frac{\partial f_0}{\partial v} = v \frac{\partial}{\partial v} \left(v f + v_0^2 \frac{\partial f}{\partial v} \right)$$

Collision as a singular perturbation

$$(u - \Omega)g(u) - \eta(u) \int_{-\infty}^{\infty} g(u') du' = -i\mu \frac{\partial}{\partial u} \left(u g + \frac{1}{2} \frac{\partial g}{\partial u} \right)$$

μ --- normalized collision frequency

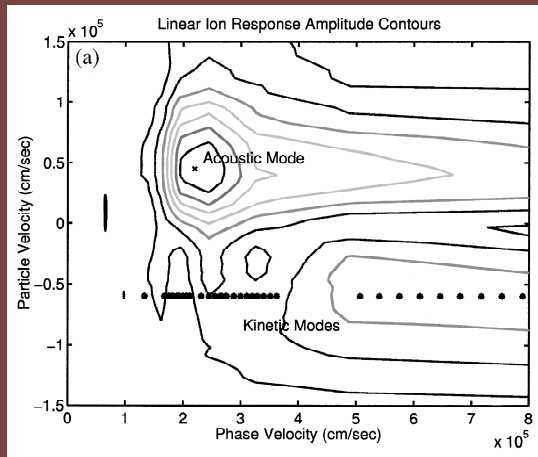
Collisional effect has to be included if there are sharp gradient in velocity space.

Each Case-Kampen mode has infinitely sharp gradient in velocity space:

$$g_{\Omega}(u) = P \left[\frac{\eta(u)}{u - \Omega} \right] + \delta(u - \Omega) \left[1 - P \int_{-\infty}^{\infty} \frac{\eta(u')}{u' - \Omega} du' \right]$$

Eigenmodes of the system change greatly even with extremely weak collision.

Discrete eigenmodes in experiments



Skiff et al. (1998)

weakly collisional

can measure distribution function accurately

Eigenmodes decay exponentially, instead of $\exp(-a\mu t^3)$ or $\exp(-a\mu x^3)$ as expected from classic theory [Su & Oberman (1968)]

A complete set of discrete eigenmodes

Ng et al., PRL, (1999, 2004)

For an initial value problem:

$$g(u,t) = \sum_n c_n g_n(u) \exp(-i\Omega_n t) \Theta(t)$$

g_n found in closed form involving incomplete gamma function.

c_n can be found by integration involving the initial data.

Similar results for boundary value problem.

Properties of the complete set of eigenmodes

The new set replaces the Case-Van Kampen modes.

It has discrete k , ω values, unlike the Case-Van Kampen modes which exist for all real k , ω .

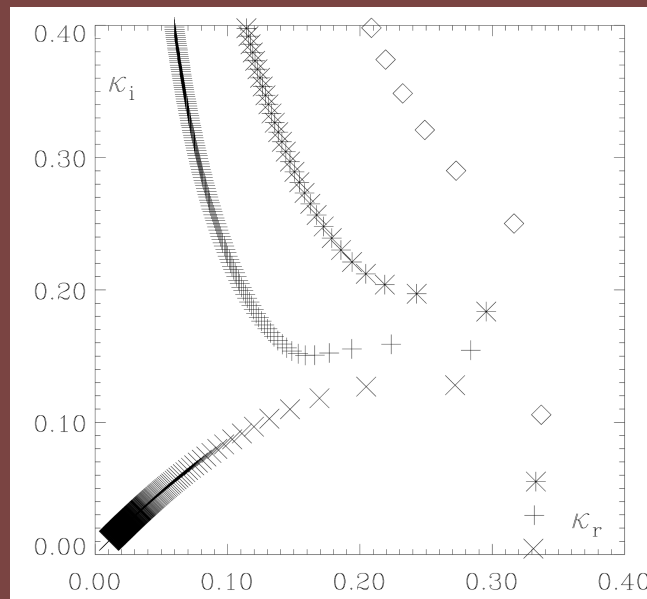
The dispersion relation tends to Landau's relation in weak collision limit:

$$D(\Omega) \xrightarrow{\mu \rightarrow 0} 1 + \alpha [1 + \Omega Z(\Omega)] = 0$$

except near the collisional modes $\Omega = -i[n\mu + 1/(2\mu)]$

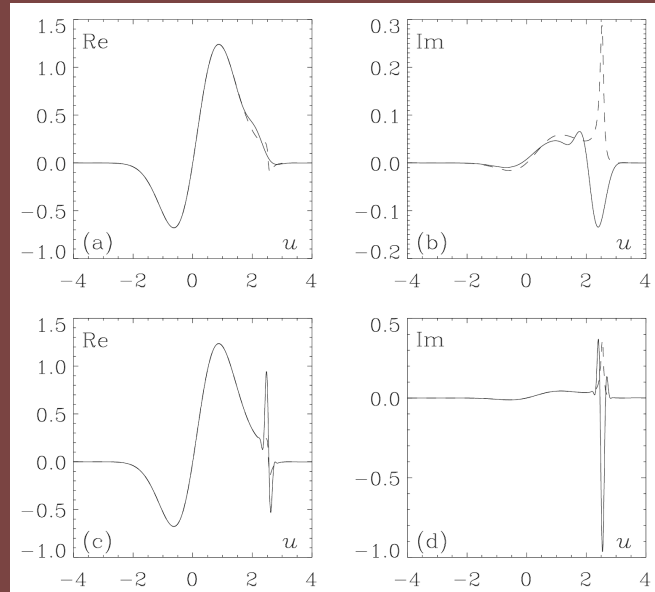
All eigenfunctions are non-singular, unlike Case-Van Kampen modes --- can excite a single mode in principle.

Calculating the eigenvalues



Eigenvalues on the complex κ plane for the spatial problem. (\diamond) $\mu = 0.1$; ($*$) $\mu = 0.05$
($+$) $\mu = 0.025$ (\times) Landau roots ($\mu = 0$).

Shape of the eigenfunctions



Solid curves: eigenfunctions for the least-damped eigenvalue Ω_0 . Dashed curves: the function $\eta(u)/(u - \Omega_0)$. (a) and (b) $\mu = 0.025$. (c) and (d) $\mu = 0.000391$.

What is a BGK mode?

An exact undamped nonlinear solution of the steady-state Vlasov-Poisson system of equations.

$$-\frac{\partial f}{\partial t} = \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \nabla \psi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0$$

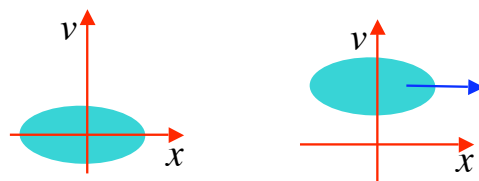
$$\nabla^2 \psi = \int d\mathbf{v} f - 1$$

Normalized; uniform ion background (for simplicity)

1D solution: *Bernstein, Greene & Kruskal (1957)*

Traveling solution by change of reference frame:

$$f(\mathbf{x} - \mathbf{v}_0 t, \mathbf{v} - \mathbf{v}_0)$$



Construction method of 1D BGK mode

BGK (1957)

$$v \frac{\partial f(x,v)}{\partial x} + \frac{d\psi(x)}{dx} \frac{\partial f(x,v)}{\partial v} = 0$$

$$\frac{d^2\psi(x)}{dx^2} = \int dv f(x,v) - 1$$

1st equation can be solved by: $f = f(w)$

$$w = v^2/2 - \psi(x)$$

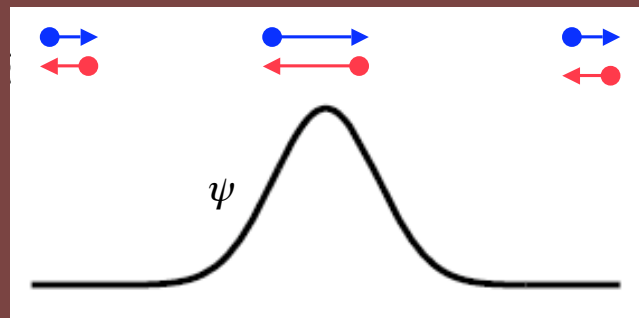
Integral-differential equation:

$$\frac{d^2\psi(x)}{dx^2} = 2 \int_{-\psi(x)}^{\infty} \frac{dw f(w)}{\sqrt{2[w + \psi(x)]}} - 1$$

Can be solved by given $f(w)$ or $\psi(x)$.

Physical meaning of 1D BGK mode

Electron velocity increases in the center, so electron density decreases



3D features in BGK mode observations

Ergun et al. (1998)

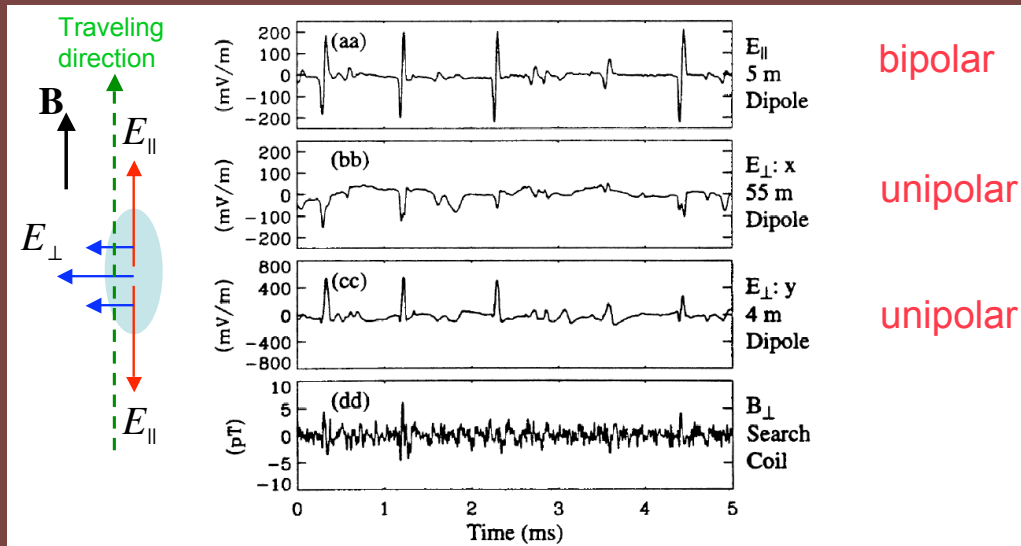


FIG. 3. Data from the Fast Auroral SnapshoT (FAST) satellite showing 3D electrostatic solitary waves observed in the auroral ionosphere. The pulses are bipolar in the parallel electric field E_{\parallel} and are unipolar in both components of perpendicular electric field E_{\perp} (from [Ergun et al. 1998]).

3D BGK mode finite \mathbf{B} ?

For infinitely strong \mathbf{B} : *Chen & Parks (2002)*
basic assumption: electrons moving along \mathbf{B} only
 \Rightarrow back to 1-D problem

Exact solutions not yet found for finite \mathbf{B} , although
approximate solutions found using drift-kinetic equation
requiring strong \mathbf{B} e.g. *Jovanovic & Shukla (2000)*

Conjecture: No solution for $\mathbf{B} = 0$? [*Chen 2002*]

Proof: conjecture is true if $f = f(w)$, i.e., f depends only on energy

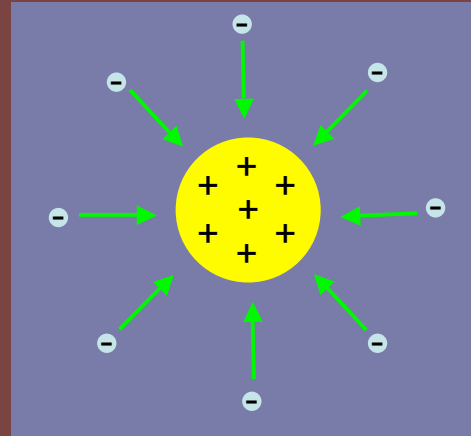
[*Ng & Bhattacharjee 2005, to be published in PRL*]

No 3D solution if f depends only on w

$$1D: \quad -\rho = \frac{d^2\psi(x)}{dx^2} = 2 \int_{-\psi(x)}^{\infty} \frac{dwf(w)}{|v|} - 1$$

$$2D: \quad -\rho = \nabla^2\psi = 2\pi \int_{-\psi}^{\infty} dwf(w) - 1$$

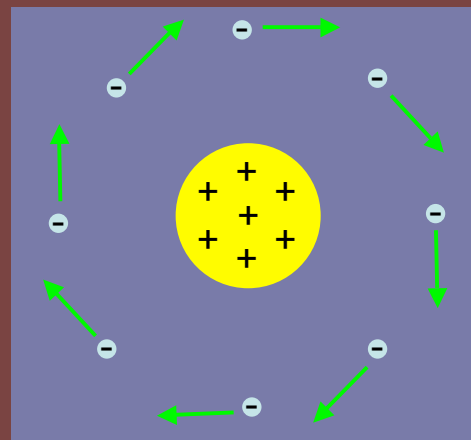
$$3D: \quad -\rho = \nabla^2\psi = 4\pi \int_{-\psi}^{\infty} |v| dwf(w) - 1$$



3D solution depending on energy as well as angular momentum

For a spherical potential $\psi = \psi(r)$
 $f = f(w, l)$ with $l = v_{\perp} r$ is also a solution of the Vlasov equation.

Possible to support the electric field self-consistently --- satisfying the Vlasov-Poisson equation.



An example BGK mode solution for $\mathbf{B} = 0$

Ng & Bhattacharjee (2005)

Particular form:
$$f(w, l) = \frac{1}{(2\pi)^{3/2}} e^{-w} f_1(l)$$

Boundary conditions: $r \rightarrow \infty$ $\psi \rightarrow 0$ $f \rightarrow \exp(-v^2/2)/(2\pi)^{3/2}$
 $f_1 \rightarrow 1$

One possible choice: $f_1(v_{\perp} r) = 1 - (1 - h_0) \exp(-v_{\perp}^2 r^2 / x_0^2)$
 $f_1(0) = h_0 \geq 0$

An example BGK mode solution for $\mathbf{B} = 0$

Poisson equation \Rightarrow
$$\frac{1}{r} \frac{d^2(r\psi)}{dr^2} = e^{\psi} h(r) - 1$$

$$h(r) = \left(h_0 + 2r^2 / x_0^2 \right) / \left(1 + 2r^2 / x_0^2 \right)$$

Boundary condition: $\psi(r \rightarrow \infty) \rightarrow 0$ $\psi(r = 0) = \psi_0$ $\psi'(r = 0) = 0$

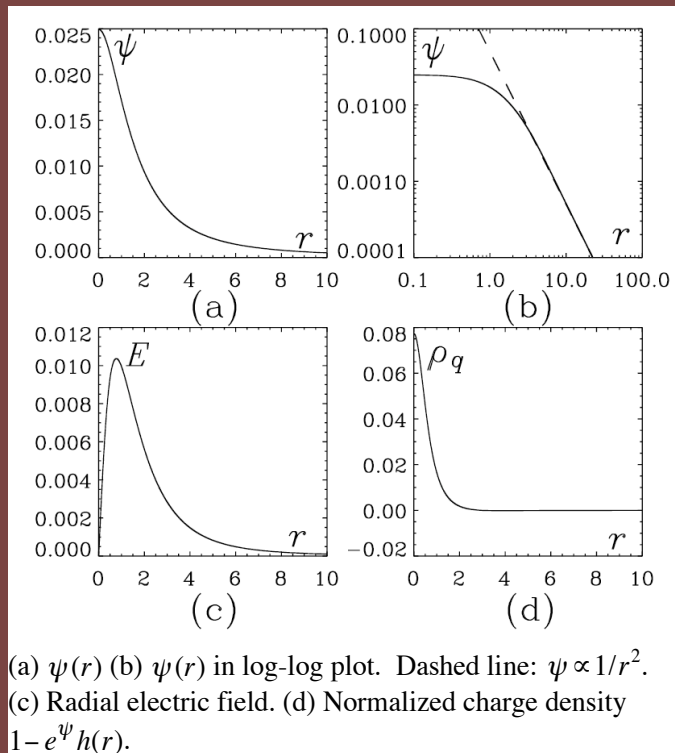
Numerical solution: solve ψ_0 for given h_0 and x_0

Numerical solution (for $0 \leq h_0 < 1$)

Electron hole

$$\psi(r \rightarrow \infty) \rightarrow \psi_\infty / r^2$$

No total net charge



3D BGK mode for finite B

Cylindrically symmetric potential: $\psi = \psi(\rho, z)$

Vlasov solution: $f = f(w, l), \quad l = 2\rho v_\phi - B\rho^2$

$$\text{Poisson eqn: } \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \psi}{\partial \rho} \right) + \frac{\partial^2 \psi}{\partial z^2} = \int d^3v f \left(\frac{v^2}{2} - \psi, 2\rho v_\phi - B\rho^2 \right) - 1$$

More difficult to solve, but may be more relevant to observations, may even be more stable.

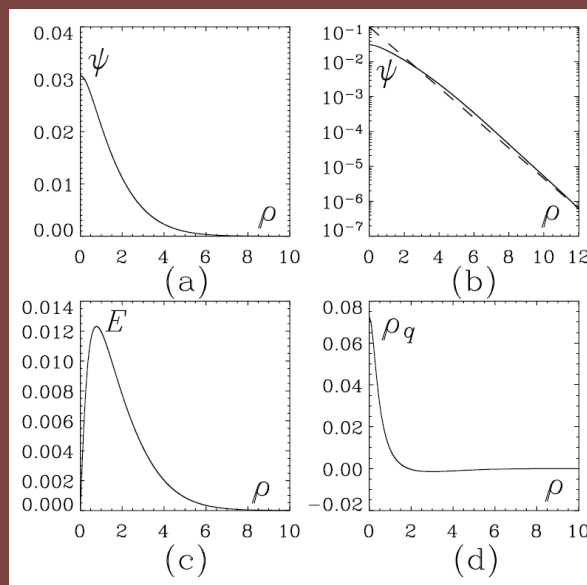
2D BGK mode for finite B

Cylindrically symmetric potential: $\psi = \psi(\rho)$

try a particular form: $f(w,l) = (2\pi)^{-3/2} \exp(-w) [1 - h_0 \exp(-kl^2)]$

Poisson equation $\Rightarrow \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{d\psi}{d\rho} \right) = e^{\psi(\rho)} \left[1 - \frac{h_0}{\sqrt{1+8k\rho^2}} \exp\left(-\frac{kB^2\rho^4}{1+8k\rho^2}\right) \right] - 1$

2D BGK mode for finite B -- Numerical solution



(a) $\psi(\rho)$ (b) $\psi(\rho)$ in semi-log plot. Dashed line: $\psi \propto \exp(-\rho)$. (c) Radial electric field. (d) Normalized charge density.

Solution for Vlasov-Poisson-Ampère equations

These solutions have finite current density due to l dependence.

Current density produces self-consistent \mathbf{B} field:

$$\nabla \times \mathbf{B} = -\beta_e^2 \int d^3v f \mathbf{v} \quad \text{with} \quad \beta_e = v_e / c$$

Solution for Vlasov-Poisson-Ampère equations found.

Magnetic field correction is small for $\beta_e \ll 1$, but can be significant even for $\beta_e \sim 0.1$

Conclusion

- ◆ In Landau damping, even a weak collision can have profound implications, and changes completely the nature of the spectrum of eigenmodes.
- ◆ A new complete spectrum of discrete eigenmodes is found that replaces the spectrum of Case-Van Kampen modes, with Landau damping rates becoming true eigenvalues.
- ◆ 2D/3D BGK modes cannot exist if the distribution function depends on energy only.
- ◆ 3D BGK modes for $\mathbf{B}=0$, and 2D BGK modes for finite \mathbf{B} are constructed when f also depends on angular momentum.

See our papers for more details:

- C. S. Ng, A. Bhattacharjee, and F. Skiff, *Phys. Rev. Lett.* **83**, 1974 (1999).
- C. S. Ng, A. Bhattacharjee, and F. Skiff, *Phys. Rev. Lett.* **92**, 065002 (2004).
- C. S. Ng and A. Bhattacharjee, *Phys. Rev. Lett.* to appear (2005).