Sufficient condition for finite-time singularity and tendency towards self-similarity in a high-symmetry flow

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Abstract

A highly symmetric Euler flow, first proposed by Kida (1985), and recently simulated by Boratav and Pelz (1994) is considered. It is found that the fourth order spatial derivative of the pressure (p_{xxxx}) at the origin is most probably positive. It is demonstrated that if p_{xxxx} grows fast enough, there must be a finite-time singularity (FTS). For a random energy spectrum $E(k) \propto k^{-\nu}$, a FTS can occur if the spectral index $\nu < 3$. Furthermore, a positive p_{xxxx} has the dynamical consequence of reducing the third derivative of the velocity u_{xxx} at the origin. Since the expectation value of u_{xxx} is zero for a random distribution of energy, an ever decreasing u_{xxx} means that the Kida flow has an intrinsic tendency to deviate from a random state. By assuming that u_{xxx} reaches the minimum value for a given spectral profile, the velocity and pressure are found to have locally self-similar forms similar in shape to what are found in numerical simulations. Such a quasi self-similar solution relaxes the requirement for FTS to $\nu < 6$. A special self-similar solution that satisfies Kelvin's circulation theorem and exhibits a FTS is found for $\nu = 2$.

On the singular blow-up of Euler,
I am an inveterate toiler;
I look near the null,
You may think this is dull,
But there's no need to be such a spoiler!

1. Introduction

Consider the three-dimensional (3D) Euler equation

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p, \tag{1}$$

with the divergence-free condition $\nabla \cdot \mathbf{v} = 0$. The self-consistent pressure p must satisfy $\nabla^2 p = -\nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v})$. An important question is whether the solution of (1) can become singular in finite time for a smooth initial condition with finite energy. Some useful and rigorous constraints on the nature of a possible finite-time singularity (FTS) have been established (Beale, Kato & Majda 1984, Constantin 1994), but the singularity has not been explicitly demonstrated in a mathematically rigorous way. Recently, an analytical model has been developed for a highly symmetric initial condition that exhibits a FTS (Bhattacharjee & Wang 1992, Bhattacharjee, Ng & Wang 1995), with some assumptions which, though physically plausible, have not yet been proved. These analytical results suggest that if an initial state with symmetries that are preserved by the Euler equation is considered, then the problem of demonstrating a FTS becomes technically less difficult. Another motivation for choosing highly-symmetric initial conditions is that they fix a geometry in place for all times, and the hope is that such studies will help identify geometrical sites where a FTS will tend to develop in less constrained flows.

A recent numerical experiment has been performed (Boratav & Pelz 1994, 1995, hereafter, BP) on a highly symmetric initial flow, first proposed by Kida (1985). The symmetries of the Kida flow are discussed in Section 2. Due to the high symmetry, BP are able to do the simulation with high spatial resolution (512^3). Within the limits of the resolution, BP report that the maximum vorticity scales as $(t_c - t)^{-1}$ as viscosity becomes small. Pelz (1997) has also proposed a vortex filament model in an effort to explain the simulation results. This model, although based on strong assumptions, predicts a FTS which evolves in a locally self-similar way near the origin. More recently, Pelz and Gulak (1997) have performed additional numerical simulations on the Kida flow. They calculate the Taylor series in time, sum the series using the Padé approximation, and recover the FTS with a value of t_c close to that found in the simulations of BP.

Ng and Bhattacharjee (1996) have proposed a sufficient condition for finite-time singularity of the Kida flow, based on analytical and numerical evidence. One way to satisfy this condition is to require that the fourth order spatial derivative p_{xxxx} be positive at the origin and within a finite range X for all time. If the range X actually tends to zero as the singularity develops, a FTS can still occur if p_{xxxx} grows rapidly enough. In Section 3, it is shown that if p_{xxxx} grows at least as fast as X^{-2}, there must be a FTS. It follows that, for a random isotropic 1-D energy spectrum $E(k) \propto k^{-\nu}$, the condition for a FTS with growing p_{xxxx} can be satisfied with the spectral index $\nu < 3$. This seems only marginally satisfied in BP's simulations where it is observed that $\nu \sim 3$.

In Section 4, we present statistical calculations, which suggest that the p_{xxxx} is very possibly positive at the origin for a random energy spectrum. This is a remarkable result since without the Kida symmetries, the expectation value of p_{xxxx} should be zero everywhere. As discussed below, this result has significant dynamical implications.

The positivity of p_{xxxx} has the intrinsic dynamical effect of favoring the development of a more coherent flow. In particular, the third order spatial derivative of the velocity u_{xxx} along the x-axis at the origin tends to decrease by the action of a positive p_{xxxx}. In Section 5, by assuming that u_{xxx} approaches the minimum allowed value for a given spectral profile, a coherent state is found which is locally self-similar around the origin. Furthermore, the velocity and pressure self-similar profile for this state is found to be similar to those observed in BP's simulations. The development of such a quasi self-similar structure is evidence supporting the realization of a FTS. Specifically, the requirement for the spectral index is now relaxed to $\nu < 6$ which appears to be consistent with simulation data. A special self-similar solution that satisfies Kelvin's circulation theorem and exhibits a FTS is found for $\nu = 2$. The time-dependence of this solution is consistent with the filament model of Pelz (1997).

2. The Symmetries of the Kida flow

The symmetries of the Kida flow have been discussed in detail by Kida (1985). Here we build these symmetries into the representations for $\mathbf{v}$ and p. The components of the velocity field $\mathbf{v} = (v_x, v_y, v_z)$ can be written as $v_x = u(x, y, z)$, $v_y = u(y, z, x)$, $v_z = u(z, x, y)$, where u can be expressed in Fourier series,

$$u(x, y, z, t) = \sum_{l,m,n} a_{lmn}(t) \sin lx \cos my \cos nz. \tag{2}$$

Here (l, m, n) are natural numbers which represent the three components of a wave vector ($l \neq 0$). In order to satisfy the symmetries and the condition $\nabla \cdot \mathbf{v} = 0$, the following conditions must hold:

$$(l, m, n) \text{ all odd or even;} \quad a_{lmn} = (-1)^l a_{lmn}; \quad \sum_{lmn}^{C} a_{lmn} = 0, \tag{3}$$

where the last summation (denoted by C) is over all permutations of any three natural numbers (l, m, n), i.e., $la_{lmn} + ma_{mnl} + na_{nlm} = 0$. By (2) and (3), it can be seen that for x close to the origin, $\mathbf{v} = O(|\mathbf{x}|^3)$. In particular, the initial state considered by both Kida (1985) and BP is $u_0 : a_{1,3,1} = 1$, $a_{1,1,3} = -1$, with all other terms set to zero. With u

represented by (2), it can be shown that the pressure p is of the form

$$p = \sum_{l,m,n} p_{lmn} \cos lx \cos my \cos nz, \tag{4}$$

where $p_{lmn} = (lA_{lmn} + mA_{mnl} + nA_{nlm})/(l^2 + m^2 + n^2)$, with A_{lmn} defined by

$$(\mathbf{v} \cdot \nabla \mathbf{v})_x = \sum_{l,m,n} A_{lmn} \sin lx \cos my \cos nz. \tag{5}$$

By (3) and (4), we know that $p_{lmn} = p_{mnl} = p_{nlm} = (-1)^l p_{lmn}$. It is easy to see that the time evolution equation,

$$\dot{a}_{lmn} + A_{lmn} - lp_{lmn} = 0, \tag{6}$$

preserves the Kida symmetries, where an overdot denotes time derivative. Equation (6) and the mode-generation scheme provide a dynamical mechanism for the excitation of modes in the small scales. From (1) and (3), we find $p_{xx}(0) = -\sum_{l,m,n} l^2 p_{lmn} = 0$, where $p_{xx}(0)$ denotes the second spatial derivative of p at the origin. This means that $p = O(|\mathbf{x}|^4)$ close to the origin and thus $p_{xxxx}(0)$ (hereinafter denoted simply by p_{xxxx}) is the first non-zero pressure derivative at the origin. Thus, the quantity p_{xxxx} has a special dynamical significance.

3. Sufficient condition for finite time singularity

Let us now consider the flow along the line $y = z = 0$. By (2), we obtain $v_y = v_z = 0$, and $v_x = u = \sum_{l,m,n} a_{lmn}(t) \sin lx$. Note that the vorticity is also identically zero along this line. This leaves open the possibility that the vortex singularity can happen infinitesimally close to this line, since the non-zero vorticity outside this line can collapse infinitesimally close to it. (This is similar to a type of FTS discussed by Bhattacharjee, Ng & Wang 1995). From (1), we obtain

$$\dot{v}_x = -p_x = \sum_{l,m,n} lp_{lmn} \sin lx(t), = -p_{xxxx}x^3/6 + O(|\mathbf{x}|^5), \tag{7}$$

where now the overdot denotes total time derivative along a fluid element moving in a trajectory $x = x(t)$ which is close to zero, with $p_{xxxx} = \sum_{lmn} l^4 p_{lmn}$. Defining $\alpha \equiv \partial_x v_x$, we obtain

$$\dot{\alpha} + \alpha^2 = -p_{xx} = \sum_{l,m,n} l^2 p_{lmn} \cos lx(t) = -p_{xxxx}x^2/2 + O(|\mathbf{x}|^4). \tag{8}$$

Note also that for the initial flow, $v_x = \alpha = 0$ at $t = 0$ for all x. From (8), we see that if $p_{xx} > 0$ for all time, then α can become negative infinite in finite time due to the presence of the α^2-term. Although this is a rigorous statement, in practice, it is difficult to calculate p_{xx} by following a fluid element. Therefore, we seek alternative formulations of this sufficient condition.

One possibility that can be shown to lead to a FTS is to assume that there exists a range $0 < x < X(t)$ in which $p_{xxxx}(x) > 0$ for all time before a possible singularity appears. (Since we will later see that $p_{xxxx}(0)$ is most probably positive for a general Kida flow, we exclude the possibility that p_{xxxx} becomes negative in finite time.) For the simplest case, we assume that $X(t) > C$, where C is a finite positive constant for all time (including $t \to t_c$). Since $p_{xxx}(0) = p_{xx}(0) = p_x(0) = 0$, it follows from the assumption above and by simple integration from $x = 0$ that the quantities $p_x(x)$, $p_{xx}(x)$, and $p_{xxx}(x)$ are also positive within the range $0 < x < X(t)$. Then by (7) and the fact that $v_x(x, t = 0) = 0$, there exists a fluid element with the Lagrangian coordinate $x(t)$ within this range $(0, C)$ that is always accelerating towards the origin $x = 0$. However, since the condition $v_x(x = 0, t) = 0$ is always maintained by the symmetry of the Kida flow, the fluid element cannot pass through the origin. The system resolves this contradiction by having the velocity derivative α blow up in finite time, since the fluid element with finite and increasing velocity is forced to go infinitesimally close to the point with zero velocity ($x = 0$). This behavior is reflected in Eq. (8) according to which α tends to negative infinity in finite time due to the presence of the α^2-term. If the time-dependence of α is determined dominantly by the α^2-term (or if the α^2-term is of the same order as the p_{xx} term), then $\alpha \to (t_c - t)^{-1}$ as $t \to t_c$. Thus, *under the assumptions discussed above, we see that the condition* $p_{xxxx} > 0$ *at the origin becomes a sufficient condition for a finite-time singularity* (Ng & Bhattacharjee 1996). This is not a point condition because it is derived under the assumption that there exists a range $X(t) > C$. Note that if we allow $X(t \to \infty) \to 0$ with a fixed value of p_{xxxx}, it is possible that $X(t)$ may shrink so rapidly that it leaves behind any trajectory before a FTS can develop by the above mechanism.

In the discussion above, we have not considered the relationship between p_{xxxx} and X. In reality, they are related since X is a measure of length scale and p_{xxxx} is found by taking spatial derivatives which should involve the inverse of the length scale. Let us assume the relation $p_{xxxx} = p_0 X^{-(2+\mu)}$, where p_0 is a positive constant. By this we have also effectively assumed that $p_{xxxx}(x)$ is a smooth function within $-X(t) < x < X(t)$ with characteristic length of the order of X. It can then be shown that:

There must be a FTS for the case with $\mu > 0$ for any smooth function $X(t)$ with continuous derivative and satisfying $X(t \neq \infty) > 0$, $X(t \to \infty) \to 0$. (A)

To show Statement (A), let us first consider the case when the speed $-\dot{X}(t)$ is a monotonically decreasing function for $t > \tilde{t} > 0$ with some constant $\tilde{t}$. By the above assumptions, we can find a finite constant $\epsilon \ll 1$ such that the Taylor approximation is good for $0 \leq x \leq \epsilon X(t)$ for all time. Then we have, in this range,

$$\dot{v}_x \approx -p_0 X^{-(2+\mu)} x^3/6. \tag{9}$$

Consider a fluid element at $x = \epsilon X(t)$ at time t such that $X(t > t' \geq 0) > x$ for all t'. Because of the initial condition $v_x(t = 0) = 0$ and the fact that this fluid element is always being accelerated toward the origin, it must have a finite negative speed. Without affecting the argument, we can treat this speed as very small compared with the speed $-\epsilon\dot{X}(t)$. It will then take a time

$$\Delta t \approx \epsilon\dot{X}/\dot{v}_x \approx -6\dot{X}/\epsilon^2 p_0 X^{1-\mu}, \tag{10}$$

for the element to be accelerated up to the same speed as $-\epsilon\dot{X}(t)$. The position ϵX will still be very close to this fluid element during this time period if

$$-\epsilon\dot{X}\Delta t \ll x = \epsilon X. \tag{11}$$

Since the magnitude of $\dot{X}(t)$ is monotonically decreasing for this case and the fluid element is accelerating, $\epsilon X(t)$ will never pass through the fluid element at any later time. Therefore the fluid element will always stay in a region in which it is always accelerated towards the origin. Since it cannot pass through $x = 0$ by symmetry, a FTS must appear to resolve the contradiction. By Eqs. (10) and (11), if the inequality

$$6X^\mu/\epsilon^2 p_0 \ll X^2/\dot{X}^2, \tag{12}$$

can be satisfied in finite time with $X(t)$ obeying the conditions stipulated in Statement (A), then there must be a FTS. It is easy to show that this is always possible when $\mu > 0$. Since if this is not so, then we must have

$$X^\mu \dot{X}^2/X^2 \equiv f^2(t) > f_0^2, \tag{13}$$

for all t, where f_0 is a finite constant. However, the solution to the equation

$$\dot{X}(t) = -f(t)X^{1-\mu/2}, \tag{14}$$

with an always positive $f(t)$ is

$$X(t) = \left[X^{\mu/2}(0) - \frac{\mu}{2}\int_0^t f(t')dt'\right]^{2/\mu}, \tag{15}$$

which must become zero in finite time and thus violate the required condition for $X(t)$ in Statement (A). (For the case with $\mu \leq 0$, the above argument does not go through and therefore a FTS does not necessarily appear.)

It is possible to generalize the proof to the case with the speed $-\dot{X}$ not monotonically decreasing. However, we will omit such a discussion here since this would take up too much space. We remark that the total time of such non-monotonic periods must be finite otherwise either a FTS will appear or X goes to zero in finite time.

From the above discussion, we see that the value of p_{xxxx} is very important to the problem of FTS, especially when it is positive and grows fast enough as the characteristic length scale of the system shrinks.

4. Positivity of p_{xxxx}

We summarize here the results obtained by Ng and Bhattacharjee (1996) that support a positive p_{xxxx}. First, we need to introduce an independent set of modes u_n of the Kida flow such that any flow satisfying the symmetries of the Kida flow can be expressed in the form

$$u = \sum_n a_n u_n, \tag{16}$$

where a_n are real constants. The choice of the set of independent modes is not unique, but we will choose to work with the simplest possible choice that each u_n consists only the Fourier modes of a given set of three all odd or all even natural number (l, m, n) that satisfy (2) and (3). All modes u_n are normalized to have the same energy.

The quantity p_{xxxx} is a quadratic function of u, so we can write $p_{xxxx} = P(u, u)$. For u expressed in (16), we obtain

$$p_{xxxx} = \sum_n a_n^2 P(u_n, u_n) + \sum_n \sum_{m \neq n} a_n a_m P(u_n, u_m) = \sum_n \sum_m a_n P_{nm} a_m, \tag{17}$$

where $P_{nm} \equiv [P(u_n, u_m) + P(u_m, u_n)]/2$. We now state a theorem:

For all the independent modes defined above, $P(u_n, u_n) > 0$.

The proof of this theorem, although straightforward, is quite complicated and needs the help of computer programs such as Mathematica to do the algebra, and thus is omitted here.

From (17) and by the theorem stated above, we see that the contributions from the self terms is always positive. Though this strengthens the case for positive p_{xxxx}, we need to consider the cross terms which cannot be neglected in principle, and they can be positive or negative depend on the sign of a_n. Let us first consider an example with two modes

$u = a_m u_m + a_n u_n$, so that $P(u,u) = a_m^2 P_{mm} + a_n^2 P_{nn} + 2a_m a_n P_{mn}$. Note that the cross term between an odd mode and an even mode is always zero. In order to have $P(u,u)$ positive, it is sufficient to have $P_{mm}P_{nn} \geq P_{nm}^2$. This condition is not always true for any two modes. However, it is also proved using Mathematica that

$$P_{nm}^2/P_{mm}P_{nn} \to 0 \text{ as } k_m^2/k_n^2 \to 0, \tag{18}$$

where k_m, k_n are the wave number of the modes u_m, u_n.

Although (18) is an relation between two modes, it can have an important effect on p_{xxxx} for a state with many modes. To see this possibility in a qualitative way, let us assume that the second term in the right hand side of (17) includes positive and negative terms in a rather random manner so that its magnitude is most probably within a range defined by the standard deviation

$$\left| \sum_n^N \sum_{m \neq n}^N a_n a_m P_{nm} \right| \sim \left[\sum_n^N \sum_{m \neq n}^N a_n^2 a_m^2 P_{nm}^2 \right]^{1/2} \sim \sum_n^N a_n^2 P_{nn}, \tag{19}$$

where the second relation is obtained by assuming that $P_{mm}P_{nn} \geq P_{nm}^2$ in the average sense because (18) holds for most terms in the double sum for large N. This means that the magnitude of the second term (which can be positive or negative) in the right hand side of (17) is most probably of the same order of magnitude of the first term, which is always positive. So the total of the two is most probably positive.

Although this is just a qualitative argument, it has been confirmed in statistical calculations. This involves a Monte Carlo calculation of p_{xxxx} as defined in (17). The energy is distributed randomly in the Fourier space according to a given energy spectrum of the form $E(k) \propto k^{-\nu}$, filling up to N modes. In most of our calculations, we choose $\nu = 3$ which is the spectrum observed by BP near $t \sim t_c$. However, the qualitative trends observed are not sensitive to variations in ν. In Fig. 1, we plot the numerical values of η, the percentage of cases with negative values of p_{xxxx} as a function of N. A straight line fit of Fig. 1 gives $\eta \sim \eta_0 (N_0/N)^{0.13}$.

The expectation value of p_{xxxx} for a random flow can be calculated directly without implementing the Monte Carlo method. In Fig. 2, values of $\langle p_{xxxx} \rangle$ as a function of k_N for different spectral indices ν are plotted. Combined with another result that $\langle X \rangle \propto k_N^{-1}$, where $\langle X \rangle$ is the expectation value of the range of positive p_{xxxx} in the x-axis, we have $\langle p_{xxxx} \rangle \propto \langle X \rangle^{-(5-\nu)}$. By Statement (A) of the previous section, we know that for the random model if $\nu < 3$, we should have a FTS.

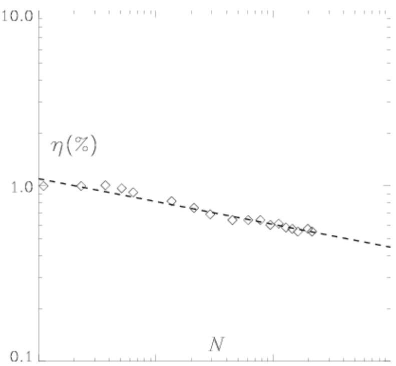

Figure 1. The possibility η (in percentage) of negative p_{xxxx} cases, as a function of the number of modes N.

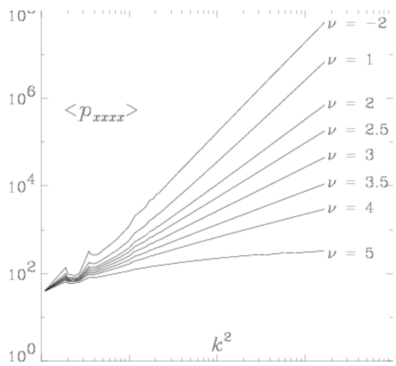

Figure 2. The expectational value $\langle p_{xxxx}\rangle$, as a function of k_N^2, for different energy spectral indices ν

For a general periodic flow, the expectation value of the pressure derivative $\langle \partial^n p/\partial x^n\rangle$ should be zero for $n \geq 1$ at all spatial point $\mathbf{x}$. The positivity of p_{xxxx} at the origin of the Kida flow is therefore quite a special case, which has important dynamical implications examined in the next section.

5. Quasi self-similarity

So far, in establishing the positivity of p_{xxxx}, only random statistics are used. In reality, the flow may evolve in a much more coherent way. In fact, BP's simulations show that the flow pattern near the origin evolves in a quasi self-similar way, which suggests a highly coherent state. In this section, we will examine the possible dynamical consequence of a positive p_{xxxx}, and how it can increase the probability of FTS formation.

Let us define u_{xxx} to be the value of $\partial^3 u/\partial x^3$ at the origin. By (7) or (8), we obtain

$$\dot{u}_{xxx} = -p_{xxxx}. \tag{20}$$

This is the first and simplest dynamical equation involving p_{xxxx}. We immediately see that this equation tends not to favor a random state since we have already establish that $\langle p_{xxxx}\rangle > 0$, over all random states. For an initial state with $u_{xxx} = 0$, it is therefore reasonable to expect that u_{xxx} will be negative for all time. Thus the true dynamics of the system tends to follow a path that is different from the random path because in the latter $\langle u_{xxx}\rangle = 0$ (This is because u_{xxx} is a linear function of the amplitudes.)

Based on the above considerations, we make a drastic assumption:

During the dynamical evolution, after a certain initial transient time, the energy is distributed among the Fourier modes such that u_{xxx} is very close to its minimal value (maximum in magnitude) for a given energy spectral profile.

Note that this assumption has not constrained the shape of the spectral profile. To simplify the discussion, we only consider spectral profiles characterized by two parameters: a spectral index ν so that the isotropic energy spectrum is given by $E(k) \propto k^{-\nu}$ up to a sharp cut off wave number K. Although a sharp cut off is physically unrealistic, this can be used to approximate the limit of a fast decay of spectral energy for high wave numbers. Note also that if $K \to \infty$ in finite time, then we must have a FTS. Furthermore, if this happens, ν has to be small enough such that the maximum vorticity and the stress tensor elements tend to infinity (Beale, Kato & Majda 1984). For a given spectral profile, it is straightforward to find the minimum u_{xxx} state. Note that because of the Kida symmetry (3), only the even modes contribute to u_{xxx}.

It is not hard to see that the flow will be locally self-similar around $\mathbf{x} = 0$, since a minimum u_{xxx} state implies definite correlation among the Fourier amplitudes. Fig. 3 plots u_x of the minimum u_{xxx} state as a function of x (in unit of π) for 26 different N from 107 to 132330, rescaled to the coordinate of the plot of $N = 107$ to show local self-similarity. We use $\nu = 2$ in these plots, but the profiles for other ν are in similar shape. Also plotted on Fig. 3 are the rescaled simulation data of BP at ($\diamond$) $t = 1.5$ and ($*$) $t = 2$ ($\sim t_c$). Similarly, Fig. 4 plots the p_{xxxx} profiles of the minimum u_{xxx} states, for N from 107 to 1624, rescaled to the coordinate of the plot of $N = 107$, with simulation data overplotted. The agreement between the simulation data and the self-similar profile of the minimum u_{xxx} state appears to be reasonably good.

For a minimum u_{xxx} state, it is straightforward to estimate that $u_{xxx} \propto -K^{(10-\nu)/2}$, and $p_{xxxx} \propto K^{8-\nu}$. By Statement (A) in Section 3, the condition for FTS is relaxed to $\nu < 6$, which appears to be quite easily satisfied, and is, in fact, consistent with the numerical results of BP. Moreover, the flow should develop a quasi self-similar form $\mathbf{v}(\mathbf{x}) \approx K^{(4-\nu)/2}\mathbf{V}(K\mathbf{x})$ for a constant ν. Since this form is so restrictive, we can actually determine the value of ν by making use of global dynamical constraints (since it may not be possible to find self-similar solution satisfying the equation of motion for any ν). Let us consider the constraint imposed by the Kelvin's circulation theorem:

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = \text{constant}, \tag{21}$$

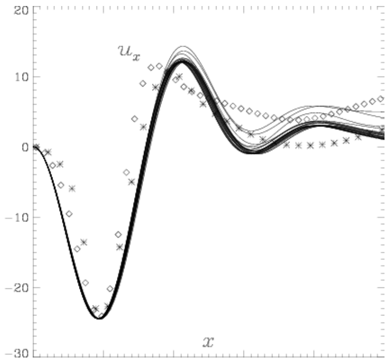

Figure 3. Plots of u_x vs. x of the minimum u_{xxx} state for N from 107 to 132330, and simulation data of BP at (⋄) $t = 1.5$ and (∗) $t = 2$, rescaled to the plot of $N = 107$.

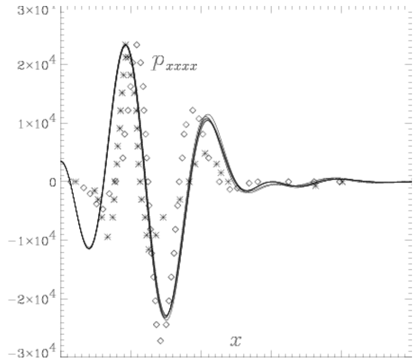

Figure 4. Plots of p_{xxxx} vs. x of the minimum u_{xxx} state for N from 107 to 1624, and simulation data of BP at (⋄) $t = 1.5$ and (∗) $t = 2$, rescaled to the plot of $N = 107$.

where C represent any close loop transported by the fluid. Substituting the self-similar form into (21), we obtain,

$$K^{(2-\nu)/2} \oint_C \mathbf{V}(\mathbf{R}) \cdot d\mathbf{R} = \text{constant}, \tag{22}$$

with $\mathbf{R} \equiv K\mathbf{r}$. Note that the loop C is being transported by the field according to the relation $\dot{\mathbf{R}} = K^{(8-\nu)/2}[\mathbf{V}(\mathbf{R}) + c\mathbf{R}]$, where we have used $\dot{K} = cK^{(8-\nu)/2}$ for $1 < \nu < 8$ with some positive constant c. Any quasi self-similar solution with a profile $\mathbf{V}(\mathbf{R})$ has to satisfy (22), besides satisfying (1) of course. This is a strong restriction on the flow. One special solution of (22) is simply obtained by choosing $\nu = 2$. This automatically makes the coefficient in (22) constant. To ensure that the loop integral remains constant also, we can choose a profile function $\mathbf{V}(\mathbf{R})$ that all the vorticity concentrate into vortex filaments. Therefore the integral for any loop that does not contain one or more filaments is zero, and for any loop that does the integral has a definite value. Moreover, the values of these integrals will not change in time. For such a quasi self-similar flow with $\nu = 2$, we obtain,

$$\mathbf{v}(\mathbf{x}) \approx K\mathbf{V}(K\mathbf{x}),\ K \approx [K_0^{-2} - Ct]^{-1/2},\ \alpha \propto K^2,\ u_{xxx} \propto K^4,\ p_{xxxx} \propto K^6. \tag{23}$$

All the scaling relations (23) agree with the results of the filament model of Pelz (1997). Note that the velocity tends to blow up in finite time.

6. Summary

We have shown that the Kida flow has a distinctive property that p_{xxxx} at the origin is most probably positive, which provides an intrinsic tendency for the flow to develop a more organized state by decreasing u_{xxx}. By simply assuming that it actually goes to the state with minimum u_{xxx}, we have found a quasi self-similar flow profile that agrees reasonably well with the simulation data. For a random energy spectrum $E(k) \propto k^{-\nu}$, it is found that the condition for the development of a FTS is to have the energy spectral index ν smaller than 6. By means of the Kelvin's circulation theorem, a simple local self-similar solution is found for $\nu = 2$ with vortex filaments.

Acknowledgments

The authors would like to thank R. B. Pelz and O. N. Boratav for providing their simulation data. This research is supported by the NSF Grant No. ATM-9801709 and the DOE Cooperative Agreement No. DE-FC02-01ER54651.

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