Inhomogeneous cyclotron emission source for the five branch mode conversion problem

C. S. Ng, V. F. Shvets, and D. G. Swanson
Physics Department, Auburn University, Alabama 36849

(Received 14 October 1994; accepted 20 January 1995)

Recent results on determining the spatial profile of the emission source distribution from a cyclotron mode conversion layer are extended to the five branch wave coupling case which occurs when the X-mode is coupled to the O-mode. A variational problem for an inhomogeneous source is formulated and solved. Computations demonstrate that the basic features, including a highly localized emitter, are generally similar to those of the three branch mode conversion problem. © *1995 American Institute of Physics.*

Recent developments in mode conversion theory as applied to spontaneous emission from inhomogeneously magnetized plasmas,[1,2] allow one to construct, by regular analytical procedures, a spatial emission source distribution from a cyclotron layer.[3] Besides obvious theoretical interest, the problem of the source distribution has clear application to the precise location of the region of the most intense cyclotron emission from magnetic fusion devices and space plasmas.

In a previous paper,[3] two fundamental points, namely the Generalized Kirchhoff's Law (GKL)[1,2]

$$E_k = A_k I_{BB}, \quad k=1,2,3, \tag{1}$$

and the principle of maximum of blackbody radiation, I_{BB}, have led, through a variational analysis, to the exact relation between a distributed absorber and an inhomogeneous emitter,

$$s(z) = w(z) \sum_{k=1}^{3} \alpha_k \Psi_k^* . \tag{2}$$

In Eq. (1), E_k and A_k are, respectively, emitted and absorbed power on the wave branches representing the k-th propagating solution of the basic three branch, fourth-order tunneling equation with localized absorption

$$\mathscr{L}(z)\psi = h(z)\Psi, \quad \Psi = \mathscr{D}\psi, \tag{3}$$

where the operators $\mathscr{L}$ and $\mathscr{D}$ are defined as

$$\mathscr{L}(z) = \frac{d^4}{dz^4} + \lambda^2 z \frac{d^2}{dz^2} + \lambda^2 z + \gamma, \; \mathscr{D} = \frac{d^2}{dz^2} + 1, \tag{4}$$

with $\lambda^2 > 0$ and $\gamma \neq -1$ both real, the wave function ψ is proportional to the transverse component of the electric field, and the imaginary part of the function $h(z)$ is responsible for cyclotron absorption and related to the absorption function $w(z)$ in Eq. (2) by $\Im h(z) = \pi\varepsilon\lambda^2 w(z) > 0$. The parameter $\varepsilon = 1 - \exp(-2\eta)$, where $\eta = \pi|1+\gamma|/2\lambda^2$ is the tunneling parameter. Equations (3) and (4) represent the simplest system of mode conversion with absorption that models a variety of physical situations [ion Bernstein wave and Alfvén waves coupling at $\omega \approx 2\omega_{ci}$, electron Bernstein wave and pure X-mode ($\omega \approx 2\omega_{ce}$), and others]. The spontaneous emission distribution function $s(z)$ is an inhomogeneous source term in the absorption–emission equation with radiative boundary conditions (outgoing waves only):

$$\mathscr{L}(z)\phi(z) = h(z)\mathscr{D}\phi + s(z). \tag{5}$$

It is clear from the GKL that given $h(z)$, $s(z)$ is not arbitrary. Moreover, it has been proved[3] that a distributed emitter is completely determined by the absorber through Eq. (2). The coefficients α_k are found from the GKL combined with the condition that the radiated power be maximized. Explicit computations performed at typical plasma parameters have demonstrated a number of qualitative differences between local emission and absorption profiles relative to those in the absence of mode conversion. In particular, a strong tendency for distributed emitters to be more highly localized than absorbers has been established in addition to a systematic shift of the maximum.

It is important to note that while the three branch mode conversion problem describes fast wave–slow wave coupling, it does not take into account the coupling between different types of fast waves which can coexist in the system. The problem in which both kinds of coupling take place is a five branch problem (two distinct fast waves propagating in either direction, and a slow wave propagating in one direction), and it has been shown[3] that any mode conversion problem in a linearly inhomogeneous magnetic field can only be either a three branch or a five branch problem. In general, the GKL is still represented by Eq. (1) but with $k=1,\ldots,K$, where K, the number of branches, is equal to 3 or 5. In this paper, we extend results obtained previously for three-wave coupling[3] to the five branch problem. The generalization is not obvious since even in the simplest cases of five branch coupling (counterpropagating X-mode and O-mode and an electron Bernstein wave near the second or third electron cyclotron harmonic) the structure of the governing equation turns out to be much more complicated. The wave equation is still of the general form of Eq. (3), but the differential operators in Eq. (3) are now higher order:

$$\mathscr{L}(z) = \frac{d^6}{dz^6} + \lambda^2 z \mathscr{D} + \gamma_2 \frac{d^2}{dz^2} + \gamma_0, \; (2\omega_{ce}), \tag{6}$$

$$\mathscr{L}(z) = \frac{d^8}{dz^8} + \gamma_6 \frac{d^6}{dz^6} - \lambda^4 z \mathscr{D} + \gamma_2 \frac{d^2}{dz^2} + \gamma_0, \; (3\omega_{ce}), \tag{7}$$

1070-664X/95/2(5)/1791/3/$6.00

where the operator $\mathscr{L}$ is now given by

$$\mathscr{L}=\frac{d^4}{dz^4}+(1+k_0^2)\frac{d^2}{dz^2}+k_0^2. \tag{8}$$

All constants in Eqs. (6)–(8) are dimensionless combinations of the electron density, n_e, temperature, T_e, resonant magnetic field, B_0, scale length of variation of the magnetic field, L, and the parallel wave number $k_\parallel$. Below, we will develop an inhomogeneous source for the X-mode–O-mode coupling case near $3\omega_{ce}$ where the absorption problem described by Eqs. (3), (7), and (8) has been considered and scattering parameters given.[4]

Defining the functional

$$\begin{aligned}P[f,g]\equiv{}& g^{(7)}\mathscr{L}f^*-g^{(6)}\mathscr{L}f'^*\\&+g^{(5)}[(1+k_0^2)f^{(4)*}+k_0^2f''^*]-k_0^2g^{(4)}f'''^*\\&+\gamma_6\{g^{(5)}\mathscr{L}f^*-g^{(4)}[(1+k_0^2)f'''^*+k_0^2f'^*]\\&+k_0^2g'''f''^*\}+\gamma_2(g''f'''^*+k_0^2g'f^*)\\&+\gamma_0\{g[f'''^*+(1+k_0^2)f'^*]-g'f''^*\}-\text{c.c.},\end{aligned} \tag{9}$$

we note that $P[f,f]$ is the energy flux which is a conserved quantity for the equation without absorption, $\mathscr{L}f=0$, and when absorption is present, the jump of $P[\psi_k,\psi_k]$ is a measure of the absorption on the k-th branch. Furthermore, the functional of Eq. (9) is related to some of the most important quantities in the source development—the matrix elements representing the scalar products[3]

$$g_{ij}\equiv\langle\Psi_i|\Psi_j\rangle_w=\int_{-\infty}^{\infty}w(z)\Psi_i^*(z)\Psi_j(z)dz=g_{ji}^*, \tag{10}$$

of solutions of the adjoint to Eq. (3). Indeed, from Eqs. (3), (7)–(10) one may obtain

$$\begin{aligned}2\pi i\lambda^4 g_{ij}&=\int_{-\infty}^{\infty}[\mathscr{L}\psi_i^*][\mathscr{K}(z)\psi_j]dz-\text{c.c.}\\&=\int_{-\infty}^{\infty}\frac{dP[\psi_i,\psi_j]}{dz}\,dz,\end{aligned} \tag{11}$$

so that g_{ij} is proportional to the jump of the functional $P[\psi_i,\psi_j]$, and the diagonal elements represent power absorbed fractions. g_{ij} can be calculated from the scattering parameters—transmission (T_i), reflection (R_i), and conversion coefficients (C_{ij})—using asymptotic expressions for linearly independent wave solutions ψ_i.[4] Straightforward but tedious analytical computations give

$$g_{ij}=\sum_{k=1}^{5}(\delta_{ik}\delta_{jk}-S_{ik}^*S_{jk})/a_k, \tag{12}$$

where $\mathsf{S}=\|S_{ik}\|$ is the matrix of scattering parameters,

$$\mathsf{S}=\begin{pmatrix}R_1 & T_1 & C_{13} & C_{14} & C_{15}\\ T_2 & R_2 & C_{23} & C_{24} & C_{25}\\ C_{31} & C_{32} & R_3 & T_3 & C_{35}\\ C_{41} & C_{42} & T_4 & R_4 & C_{45}\\ C_{51} & C_{52} & C_{53} & C_{54} & R_5\end{pmatrix}, \tag{13}$$

and $\mathbf{a}=(\rho_{xs},\rho_{xs},\rho_{os},\rho_{os},1)$, where the ρ-constants are given in terms of the original dimensionless parameters[4] by $\rho_{xs}=1/\rho_{sx}=g_1g_2/\varepsilon_1$, $\rho_{os}=1/\rho_{so}=g_2/\varepsilon_1$, $\varepsilon_k=1-g_k$, with

$$g_1=\exp[2\pi(1+\gamma_0-\gamma_2-\gamma_6)/2\lambda^4(1-k_0^2)],$$

$$g_2=\exp\{2\pi[\gamma_0-k_0^2\gamma_2+k_0^6(k_0^2-\gamma_6)]/2\lambda^4k_0(k_0^2-1)\}.$$

From Eqs. (12), (13), and the expressions for power absorbed fractions in terms of the scattering parameters,[4] the power absorbed fractions are given by

$$A_k=a_kg_{kk}=a_k\int_{-\infty}^{\infty}w(z)|\Psi_k(z)|^2dz,\quad k=1,\ldots,5. \tag{14}$$

To find general expressions for the emissivities, E_k, we extend the previously obtained[5] three branch Green function solution for the emission problem represented by Eqs. (5), (7), and (8). The solution of this problem can be written in the form

$$\phi(z)=\int_{-\infty}^{\infty}s(y)\sum_i B_i(z)\Psi_i(y)dz, \tag{15}$$

where the summation is taken over $i=2,4,5,8$ as $z>y$ and $i=1,3,6,7$ as $z<y$. Proceeding as before,[3] and considering asymptotic expansions of the functions $B_i(z)$ as $z\to\pm\infty$, one obtains the following expressions for emitted energy fractions carried by the outgoing waves:

$$E_k=a_k\left|\int_{-\infty}^{\infty}s(z)\Psi_k(z)dz\right|^2,\quad k=1,\ldots,5. \tag{16}$$

The results (14) and (16), along with the GKL,

$$\frac{E_1}{A_1}=\frac{E_2}{A_2}=\frac{E_3}{A_3}=\frac{E_4}{A_4}=\frac{E_5}{A_5}=I_{BB}, \tag{17}$$

lead to the variational problem for determining $s(z)$: all E_k are functionals of s which must give the same maximum value to each ratio in Eq. (17) to satisfy the condition of the maximum of blackbody radiative power. Further, $s(z)$ is bounded in magnitude, and it was shown previously[3] that the corresponding integral condition is of the form:

$$\int_{-\infty}^{\infty}\frac{|s(z)|^2}{w(z)}\,dz=P_0. \tag{18}$$

The normalizing constant P_0 is unknown in advance (except that $P_0\geqslant I_{BB}$ from the Cauchy–Schwartz inequality) but will be determined in the process of solution. To maximize five functionals $I_k[s]=E_k[s]/A_k$ simultaneously, with the additional GKL and normalization constraints, we should vary the functional

$$\Phi[s]=P_0[s]-\sum_{k=1}^{5}\mu_k(I_k[s]-I_{BB}), \tag{19}$$

where $P_0[s]$ is given by Eq. (18) and the μ_k are Lagrange multipliers. This functional has zero variation if $s(z)$ obeys the integral equation

$$s(z)=w(z)\int_{-\infty}^{\infty}K(z,y)s(y)dy, \tag{20}$$

with the kernel

$$K(z,y)=\sum_{k=1}^{5}\mu_k\Psi_k^*(z)\Psi_k(y)/g_{kk}.$$

Equation (20) has a solution of the form

$$s(z)=w(z)\sum_{k=1}^{5}\alpha_k\Psi_k^*, \tag{21}$$

where the coefficients

$$\alpha_k=\frac{\mu_k}{g_{kk}}\int_{-\infty}^{\infty}\Psi_k(y)s(y)dy \tag{22}$$

are still to be determined. Introducing the matrices $\mathsf{Y}=\|Y_{kj}\|,Y_{kj}=g_{jk}/\sqrt{g_{jj}g_{kk}}=Y_{jk}^*$ and $\mathsf{R}=\mathsf{Y}^{-1}$, and then changing variables,

$$\omega_k=\sum_{j=1}^{5}\sqrt{\frac{g_{jj}}{I_{BB}}}Y_{kj}\alpha_j, \tag{23}$$

it follows from Eqs. (17), (18), (21), and (22) that the problem is reduced to minimizing the quadratic form

$$\boldsymbol{\omega}^\dagger\mathsf{R}\boldsymbol{\omega}=P_0/I_{BB},\boldsymbol{\omega}=\{\omega_j\},\ j=1,\ldots,5 \tag{24}$$

with the GKL constraints

$$|\omega_1|=|\omega_2|=|\omega_3|=|\omega_4|=|\omega_5|=1. \tag{25}$$

Using these, we can write $\omega_j=\exp(i\theta_j)$ and arrive, through Eq. (24), at an unconstrained minimization problem for

$$f(\theta)=\sum_j R_{jj}+\sum_{j\neq k}R_{jk}e^{i(\theta_k-\theta_j)}. \tag{26}$$

Clearly, one of the θ_j is arbitrary (e.g., $\theta_1=0$) since only relative values appear in Eq. (26). This reduces the problem to minimizing a real function of four real unconstrained variables. In our computations, we used the downhill simplex method in multidimensions, described e.g., in Ref. 6. After finding the θ_j, all the α_k are given by

$$\alpha_k=\sum_{j=1}^{5}\sqrt{\frac{I_{BB}}{g_{kk}}}R_{kj}e^{i\theta_j}, \tag{27}$$

so that the emission source distribution from a cyclotron layer is now completely determined by Eq. (21). The normalizing constant P_0 is now determined as well by Eqs. (23) and (24).

Using the technique developed above, we performed a number of computations near the third electron cyclotron harmonic with X-mode–O-mode coupling. Our general conclusion is that while mathematical difficulties of source development are much greater for the five branch coupling as compared to a three branch problem, the basic influences of mode conversion on the shape of the inhomogeneous emission source do not change qualitatively. This can be explained by the relatively weak coupling of the O-mode with the other wave branches. In particular, one of the important features—higher localization of the emitter in comparison with the absorber—has been confirmed for five branches as well. In the example depicted in Fig. 1, where emission and absorption profiles are drawn together (each normalized to unit maximum), this tendency is clearly seen. The narrow character of the emissivity distribution is important since it facilitates the accurate location of radiative regions in inhomogeneous plasmas which are necessary to interpret emission data. While the shift is only about a half centimeter in the case illustrated, the scale length is only 10 cm, which is about a 5% shift. The extension of the absorption and emission functions beyond the origin, which is absent when $n_\parallel=0$, is due to the finite value of $n_\parallel$, and the zero crossing for $w(x)$ derives from the same fact, since the source/sink function was derived from the relativistic function $\mathscr{F}_{9/2}(z,a)$ rather than the pure X-mode function $F_{9/2}(z)$, where in this example $a=0.511$. Only the magnitude of the source distribution function is shown, so the emission is of course positive everywhere.

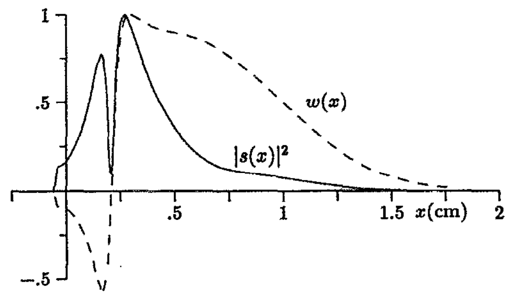

FIG. 1. Electron absorption function, $w(x)$, and distributed source strength, $|s(x)|^2$, with $n_e=2.7\times10^{17}\ \mathrm{m}^{-3}$, $B_0=0.1$ T, $L=0.1$ m, $T_e=5$ keV, and $n_\parallel=0.1$.

ACKNOWLEDGMENT

This work was supported by the U.S. Department of Energy under Contract No. DE-FG05-85ER53206-94.

[1]D. G. Swanson and S. Cho, Phys. Rev. Lett. **63**, 1386 (1989).
[2]D. G. Swanson and V. F. Shvets, Phys. Rev. Lett. **68**, 3036 (1992).
[3]V. F. Shvets and D. G. Swanson, J. Plasma Phys. **50**, 163 (1993).
[4]J. L. Hu and D. G. Swanson, Phys. Fluids B **5**, 4221 (1993).
[5]S. Cho and D. G. Swanson, Phys. Fluids B **2**, 2704 (1990).
[6]W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes: the Art of Scientific Computing* (Cambridge University Press, Cambridge, 1989).