Three-dimensional Bernstein-Greene-Kruskal modes in a multi-species plasma: Void solutions in a dusty plasma?

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Abstract

A recent theory on three-dimensional (3D) Bernstein-Greene-Kruskal (BGK) mode [Ng and Bhattacharjee, Phys. Rev. Lett., 95, 245004 (2005)] is generalized to the case of a multi-species plasma. One particular class of exact Vlasov solutions is sought and some are constructed explicitly with electrons and ions following Boltzmann distributions and the distribution function of the other species depending on energy and angular momentum. Some of these solutions are shown to have a depleted (void) or enhanced (anti-void) density of the third species, or even more complex structures such as a void shell. Comparison with dust voids observed in dusty plasma experiments will be made, as well as a recent theory of the dynamical formation of dust voids [Avinash, Bhattacharjee and Hu, Phys. Rev. Lett., 90, 075001 (2003)], which has been generalized to 3D [Ng et. al., Phys. Plasmas, submitted].

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**Introduction**

◊ In recent years, many experiments in dusty plasmas reported the formation of voids (regions within a sharp boundary that have no dust particles within).

◊ Theoretical understanding of void formation is still limited and sometimes controversial.

◊ BGK modes in a dusty plasma have been used as a possible model for a dust void [e.g., *Jovanovic and Shukla 2003*].

◊ Full 3D/2D exact analytic solutions of BGK modes have been constructed recently [Ng and Bhattacharjee 2005, Ng, Bhattacharjee, and Skiff 2006], which depend not only on energy, but also on angular momentum.

◊ Such solutions can be generalized to the case with a multi-species plasma, e.g., a dusty plasma, if effects of collision are neglected. We show that these solutions can support many different classes of dust density profiles, including a dust void.
Experimental observations of dust voids

[From D. Samsonov and J. Goree, Phys. Rev. E 59, 1047 (1999).]

Dust voids in numerical fluid simulations

2D simulation of a void formation fluid model as the saturated state of an instability caused by ion drag.

[From Ng et. al., Phys. Plasmas, submitted.]
Dust voids in numerical fluid simulations

Comparisons of the radial profiles from a 2D simulation (solid) and a 1D radial simulation (dash-dotted).

[From Ng et. al., Phys. Plasmas, submitted.]
Vlasov equation

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} + \frac{q_s}{m_s} \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0,$$

i.e., the Boltzmann’s equation without collision. Couple with the Maxwell’s equations:

$$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E}, \quad \nabla \cdot \mathbf{B} = 0,$$

$$\nabla \cdot \mathbf{E} = 4\pi \rho = 4\pi \sum_s q_s \int d\mathbf{v} f_s,$$

$$\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi \mathbf{J}}{c} = \frac{4\pi}{c} \sum_s q_s \int d\mathbf{v} \mathbf{v} f_s.$$
Exact static solution

\[ \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} - \frac{q_s}{m_s} \nabla \psi \cdot \frac{\partial f_s}{\partial \mathbf{n}} = 0 , \]

\[ \nabla^2 \psi = -4\pi \sum_s q_s \int d\mathbf{v} f_s , \]

\[ \mathbf{E} = -\nabla \psi . \]

A BGK solution is an exact nonlinear solution of these equations.
3D BGK mode --- electron hole

\[ \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \nabla \psi \cdot \frac{\partial f}{\partial \mathbf{r}} = 0 \]  
--- Vlasov equation

\[ \nabla^2 \psi = \int d\mathbf{v} f - 1 \]  
--- Poisson equation

Normalization procedure: \( \mathbf{v} \rightarrow v_e \mathbf{v}, \mathbf{r} \rightarrow \lambda \mathbf{r}, \psi \rightarrow 4\pi n_0 e \lambda^2 \psi \), and \( f_e \rightarrow n_0 f / v_e^3 \).

\( v_e \) : electron thermal velocity
\( \lambda = v_e / \omega_{pe} \) : electron Debye length

The form

\[ f = f(w,l) \]

solves the Vlasov equation exactly, where \( l = v_\perp r \) is the angular momentum. There can be many different forms of \( f \) within this general form that satisfy the Poisson equation also.
3D spherically symmetric case: example

Consider the form,
\[
f(w,l) = \frac{1}{(2\pi)^{3/2}} e^{-w} f_1(l).
\]

Local solution requires b. c. of
\[
\psi \to 0, \ f \to \exp(-v^2/2)/(2\pi)^{3/2}, \text{ or } f_1 \to 1,
\]
as \(r \to \infty\).

E. g.,
\[
f_1(v \perp r) = 1 - (1 - h_0) \exp(-v^2 r^2 / x_0^2)
\]
So,
\[
\frac{1}{r} \frac{d^2 (r \psi)}{dr^2} = e^\psi h(r) - 1,
\]
\[
h(r) = \left(h_0 + 2r^2 / x_0^2\right) \left(1 + 2r^2 / x_0^2\right),
\]
subject to the b. c.: \(\psi(r \to \infty) \to 0\), \(\psi(r = 0) = \psi_0\), and \(\psi'(r = 0) = 0\).
(a) Numerical solution $\psi(r)$ for the case with $h_0 = 0.9$ and $x_0 = 1$. (b) The same solution in log-log plot. The dashed line showing $\psi_\infty = x_0^2 (1 - h_0) / 2$. (c) Radial electric field. (d) Normalized charge density $1 - e^{\psi} h(r)$. 

Numerical solution \(\text{for } 0 \leq h_0 < 1\)
Multi-species plasma --- Vlasov solutions

For simplicity, consider one ion species and electrons both with Boltzmann distributions:

\[ f_i = \frac{n_{i0}}{(2\pi kT_i/m_i)^{3/2}} \exp\left( \frac{-Z_i e \psi - m_i v^2 / 2}{kT_i} \right), \]
\[ f_e = \frac{n_{e0}}{(2\pi kT_e/m_e)^{3/2}} \exp\left( \frac{e \psi - m_e v^2 / 2}{kT_e} \right). \]

Add a third species, e.g., dust particles, with mass \( m_d \), and charge \( -Z_d e \), with a distribution of the form

\[ f_d = f_d(w, l) = f_d(Z_d e \psi - m_d v^2 / 2, v \perp r). \]

The Poisson equation in dimensionless form becomes

\[ \frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \right) = n_{e0} \exp(\beta_e \psi) - e^\psi + n_d, \]

where \( r \) is in units of \( (kT_i/4\pi n_{i0})^{1/2} / Z_i e \), \( \psi \) is in units of \( kT_i/Z_i e \), \( n_{e0} \) is in units of \( Z_i n_{i0} \), \( n_d \) is in units of \( Z_i n_{i0} / Z_d \), \( \beta_e = T_i / Z_i T_e \), \( \beta_d = Z_d T_i / Z_i T_d \), and

\[ n_d = \int f_d(w, l) d^3v. \]

Therefore, different form of \( f_d \) will produce different profiles of \( n_d \), after solving \( \psi \).
Void --- solution 1

\[ f_d = \frac{n_{d0}}{(2\pi kT_d/m_d)^{3/2}} \exp\left(\frac{Z_e e \psi - m_d v^2/2}{kT_d}\right) \left[ 1 - \exp\left(-\frac{m_d v^2 r^2}{2kT_d x_0^2}\right) \right], \]

such that by integration, in dimensionless form,

\[ n_d = n_{d0} \exp(\beta_d \psi) \left[ \frac{r^2}{r^2 + x_0^2} \right]. \]

\[ \beta_e = 1, \quad \beta_d = 100, \quad x_0 = 1, \quad n_{e0} = 0.1, \quad n_{d0} = 1 - n_{e0} = 0.9. \]
Void --- solution 1
Void --- solution 2

\[ f_d = \frac{n_{d0}}{(2\pi kT_d/m_d)^{3/2}} \exp \left( \frac{Z_d e \psi - m_d v^2 / 2}{kT_d} \right) \left\{ 1 - \exp \left[ -\left( \frac{m_d v_{1r}^2 r^2}{2kT_d x_0^2} \right) \right] \right\} \]

such that by integration, in dimensionless form,

\[ n_d = n_{d0} \exp(\beta_d \psi) \left[ 1 - \frac{x_0^2 \sqrt{\pi}}{2r^2} \text{erfc}\left( \frac{x_0^2}{2r^2} \right) \exp\left( \frac{x_0^4}{4r^4} \right) \right]. \]

\[ \beta_e = 1, \quad \beta_d = 100, \quad x_0 = 1, \quad n_{e0} = 0.1, \quad n_{d0} = 1 - n_{e0} = 0.9. \]
Void --- solution 2
Void --- solution 3

\[ f_d = \frac{n_{d0}}{(2\pi kT_d/m_d)^{3/2}} \exp\left(\frac{Z_d e \psi - m_d v^2/2}{kT_d}\right) \Theta\left(\frac{m_d v^2 r^2}{2kT_d x_0^2} - 1\right) \]

such that by integration, in dimensionless form,

\[ n_d = n_{d0} \exp\left(\beta_d \psi - \frac{x_0^2}{r^2}\right). \]

\beta_e = 1, \beta_d = 100, x_0 = 1, n_{e0} = 0.1, n_{d0} = 1 - n_{e0} = 0.9.
Void --- solution 3
Void --- solution 3

\[ n_{d} \]

\[ n_{d} \]

\[ \beta_{e} = 1, \]
\[ \beta_{d} = 100, \]
\[ x_{0} = 2, \]
\[ n_{e0} = 0.5, \]
\[ n_{d0} = 0.5. \]
Void --- solution 4

\[ f_d = \frac{m_d n_d^0}{\pi r} \Theta(r - x_0) \delta \left( m_d \frac{v_r^2}{r} + Z_d e \frac{d\psi}{dr} \right) \delta(v_r) , \]  

\[ \frac{d\psi}{dr} < 0 \] wherever \( n_d \neq 0 \), such that by integration,

\[ n_d = n_d^0 \Theta(r - x_0) . \]

\[ \beta_e = 1, \beta_d = 100 , \ x_0 = 1, \ n_{e0} = 0.1, \ n_{d0} = 1 - n_{e0} = 0.9 . \]
Void --- solution 4
**Void shell**

\[ f_d = \frac{m_d n_{d0}}{\pi r} \left[ \Theta(r - x_0) - \Theta(r - x_1) + \Theta(r - x_2) \right] \delta \left( m_d \frac{v^2}{r} + Z_d e \frac{d\psi}{dr} \right) \delta(v_r) \]

with \( \frac{d\psi}{dr} < 0 \) wherever \( n_d \neq 0 \), such that by integration,

\[ n_d = n_{d0} \left[ \Theta(r - x_0) - \Theta(r - x_1) + \Theta(r - x_2) \right]. \]

\[ \begin{array}{c|c}
\hline
r & n_d \\
\hline
0 & 1.0 \\
2 & 0.8 \\
4 & 0.6 \\
6 & 0.4 \\
8 & 0.2 \\
10 & 0.0 \\
\hline
\end{array} \]

\[ \beta_e = 1, \ \beta_d = 100, \ x_0 = 1, \ n_{e0} = 0.1, \ n_{d0} = 0.9, \ x_1 = 1.4, \ x_2 = 1.8. \]
Void shell
Shell-shape modulations

\[
f_d = \frac{n_{d0} \exp\left(\frac{Z_d e \psi - m_d v^2 / 2}{kT_d}\right)}{\left(2\pi kT_d / m_d\right)^{3/2}} \left[1 - \alpha_0 \Theta\left(\frac{m_d v^2 r^2}{2kT_d x^2} - 1\right) + \alpha_1 \Theta\left(\frac{m_d v^2 r^2}{2kT_d x^2} - 1\right)\right]
\]
such that by integration,

\[
n_d = n_{d0} e^\beta_{\psi}\left[1 - \alpha_0 \exp\left(-\frac{x_0^2}{r^2}\right) + \alpha_1 \exp\left(-\frac{x_1^2}{r^2}\right)\right].
\]

\[
\beta_e = 1, \quad \beta_d = 100, \quad x_0 = 1, \quad x_1 = 400, \quad n_{e0} = 0.82, \quad \alpha_0 = 1, \quad \alpha_1 = 0.2
\]
\[
n_{d0} = \frac{(1 - n_{e0})}{(1 - \alpha_0 + \alpha_1)} = 0.9.
\]
Shell-shape modulations

\[ \psi \]
\[ \frac{d\psi}{dr} \]
\[ |\psi| \]
\[ n_e \]
\[ n_i \]
\[ \rho \]
Anti-void

\[ f_d = n_{d0} \exp \left( \frac{Z_d e \psi - m_d v^2 / 2}{k T_d} \right) \left[ 1 - \alpha_0 \Theta \left( \frac{m_d v^2 r^2}{2 k T_d x_0^2} - 1 \right) \right], \]

such that by integration,

\[ n_d = n_{d0} e^{\beta_d}\psi \left[ 1 - \alpha_0 \exp \left( -\frac{x_0^2}{r^2} \right) \right]. \]

\[ \beta_e = 1, \ \beta_d = 100, \ x_0 = 1, \ n_e_0 = 0.9991, \ \alpha_0 = 0.999, \]
\[ n_{d0} = \frac{(1 - n_{e0})}{(1 - \alpha_0)} = 0.9. \]
Anti-void
Conclusion

◊ Exact 3D BGK modes in a multi-species plasma are constructed explicitly.

◊ With electrons and ions following Boltzmann distributions and the distribution function of the other species depending on energy and angular momentum, the density profile of the third species can have many different structures, including voids, void shells, and anti-voids.

◊ Even more complex structures should exist, especially if ions or electrons are not simply following Boltzmann distributions.

◊ These solutions may be applicable to dusty plasmas, if effects of collision are neglected, or to multi-species plasmas with more than one type of ions. Experimental verifications of these 3D BGK modes would be interesting.

◊ Unlike a recent dust void formation theory [Avinash, Bhattacharjee and Hu 2003; Ng et. al 2006], the formation and dynamical accessibility of such void solutions is not considered. The physical mechanisms to maintain the final steady state structure in both theories are also very different. However, a complete void theory may need to account for both types of effects.