Anisotropic MHD Turbulence in the Interstellar Medium and Solar Wind

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Abstract. A theoretical model is given of anisotropic magnetohydrodynamic turbulence in the interstellar medium and the solar wind. The model is motivated by observations that show significant deviations from the Kolmogorov power-law. Dimensional and heuristic arguments are given and critically assessed. On the basis of the weak turbulence approximation in which three-wave interactions dominate, analytical and numerical results are obtained for the anisotropic energy spectrum produced by the random scattering of shear Alfvén waves propagating parallel to a large-scale magnetic field. The energy spectrum is shown to be proportional to $k^{-2}$, qualitatively consistent with some observations and wave kinetic theory.

1. INTRODUCTION

Our observational knowledge of the small-scale density fluctuations in the ionized interstellar medium (ISM) is primarily due to interstellar scintillations [1,2]. These observations show two qualitatively important features of density fluctuation spectra: they obey power laws and are anisotropic. However, some delicate observational and theoretical issues complicate precise quantitative results on the power-law exponent(s) and the degree of anisotropy.

If we write the power spectrum in the form

$$P_P(k) = C_\nu^2 k^{-\alpha} \quad k_{in} \leq k \leq k_{out},$$

where $k = |\mathbf{k}|$ is the magnitude of the wave number and $C_\nu^2$ is a positive constant, a large number of observations report that the exponent $\alpha$ is approximately equal to $11/3$ over many decades in $k$ [3-7]. Equivalently, if the power spectrum is expressed as a one-dimensional form in $k$-space, the exponent is given by $\beta = \alpha - 2 = 5/3$ which is identical to that predicted by Kolmogorov’s well-known inertial range spectrum for turbulent fluids [8]. Although this is suggestive [3,9], it is far from obvious why Kolmogorov’s spectral law for an incompressible and isotropic neutral fluid should apply at all to the ISM which is a compressible ionized medium permeated by a large-scale and directed magnetic field.

A definitive theoretical interpretation of the observed spectra is difficult for at least two reasons. First, the exponent $\alpha$ (or $\beta$) depends sensitively on the mechanism that produces the turbulence and needs to be determined with a high level of precision in order to discriminate between different theoretical models. A number of observations show that $\alpha(\beta)$ is less than 4(2), but this is not precise enough to discriminate between different theoretical models. Second, the claim that the power-law should be attributed to an inertial range spectrum carries with it not only the challenge of establishing the power law over several decades in $k$, but also the identification of an outer wave number $k_{out}$ where energy is injected and an inner wave number $k_{in}$ above which energy is dissipated. The realization of both of these objectives simultaneously by independent measurements is difficult. There has been a general tendency among observers to settle for the Kolmogorov exponent $\beta = 5/3$ [10-14]. However, a significant number of observations show deviations from Kolmogorov scaling [13,15-18]. The error bars in these observations are sufficiently small that they raise questions regarding the universal validity of a Kolmogorov scaling for ISM turbulence. In fact, these examples suggest that $\beta$ lies in the range between 1.9 and 2. Lambert and Rickett [18] have shown recently that many features of diffractive measurements can be accounted for by a non-turbulent $\beta = 2$ model with abrupt (or discontinuous) changes in the density profile of the ISM [19], but due to the presence of several discrepant features in the data they rule out both the $\beta = 2$ model as well as the Kolmogorov model as universal models for ISM fluctuations.
They, therefore, suggest the development of different spectral models for different lines of sight.

As mentioned above, there is significant observational evidence to suggest that the interstellar scintillation spectrum is anisotropic [6,13,14,20]. The density irregularities have a cigar-like structure, with long spatial scales parallel and short spatial scales perpendicular to the background field [21]. The degree of anisotropy is different for different sources (that is, different lines of sight through the ISM). Averaging along the line of sight can cause a reduction in the measured degree of anisotropy.

Non-Kolmogorov power laws for velocity and magnetic field fluctuation spectra have also been observed in the turbulent solar wind. Observations from Voyager 1 and 2 spacecrafts between 13 AU and 25 AU show $k^{-2}$ spectra at low frequencies at low heliographic latitudes [22,23]. A possible explanation of this spectrum is that it is mainly due to the presence of shocks and discontinuities [24,25]. However, it is also possible that the spectra may have a turbulent origin, with turbulence eventually steepening to produce shocks. As in the ISM, anisotropy is a persistent feature of solar wind turbulence and manifests itself in several in situ observations as more power perpendicular than parallel to the local magnetic field [26-28].

In this paper, we present a new calculation of the anisotropic energy spectrum in a plasma permeated by a uniform background magnetic field. The closure employed in this calculation is weak turbulence [29,30] which has been shown to be dominated by three-wave interactions [31-33]. We show by means of a novel simulation of the random scattering of shear Alfvén waves that the inertial range anisotropic energy spectrum is proportional to $k_{⊥}^{-2}$, obtained earlier by heuristic analysis [34,35] and wave kinetic theory [33]. Although the geometry of our model is simple and weak turbulence closure is restrictive, the calculation provides qualitative support for some of the observations on non-Kolmogorov power-laws.

The following is a layout of this paper. In §2, we review the dimensional and heuristic arguments for the Kolmogorov, Iroshnikov-Kraichnan (IK) [36,37] and anisotropic MHD energy spectra. We do so because although such arguments have been successful and are widely used, they can be problematic, reinforcing the need for careful dynamical calculations. In §3, we present analytical and numerical calculations that test and verify the heuristic arguments for anisotropic weak MHD turbulence. We conclude in §4 with a summary of our results, a discussion of the limitations of our theoretical model, and implications for observations of turbulence in the ISM and the solar wind.

2. DIMENSIONAL AND HEURISTIC ANALYSIS

Kolmogorov derived his celebrated energy spectrum for hydrodynamics (HD) essentially by dimensional analysis [8]. He made two crucial assumptions: the turbulence is isotropic and the dominant interactions between eddies are local in $k$-space.

If the turbulence is isotropic in $k$-space, the energy can be written $\int E(k)dk$ where $E(k)$ is the energy spectrum. Assume, following Kolmogorov, that there exists an inertial range such that the energy transfer rate $\varepsilon(k)$ is a constant independent of $k$ and furthermore, that the energy transfer process is local in $k$-space. Since the dimension of $\varepsilon$ is $LT^{-3}$, $k$ is $L^1$ and $E(k)$ is $LT^{-2}$ (where $L$ is the dimension of length and $T$ is of time), dimensional homogeneity of the relation $\varepsilon \sim k^a E_k^p$ yields the inertial-range energy spectrum $E(k) \propto \varepsilon^{2/3} k^{-5/3}$.

IK extend Kolmogorov’s analysis to incompressible magnetohydrodynamic (MHD) turbulence. As discussed by Kraichnan [37], the small wave number components act like a background magnetic field which cannot be removed by a Galilean transformation and support Alfvén wave packets propagating in both directions with the Alfvén speed $V_A$. An Alfvén wave packet can interact with another wave packet only if the two collide, with the interaction time given typically by $\tau_c \sim (k V_A)^{-1}$. Note that $\tau_c$ has an inherently nonlocal character in $k$-space because it depends not only on the typical spatial dimension ($k^{-1}$) of a wave (a local property) but also on the magnitude of the large-scale (small $k$) magnetic perturbation that determines $V_A$ (a nonlocal property). In many cases of physical interest $\tau_c \sim (k V_A)^{-1}$ is much shorter than the eddy turnover time $(k V_A)^{-1}$, with the consequence that the energy cascade is more inhibited in MHD than it is in HD. By treating the $k=0$ component of the magnetic field at any spatial location as the background uniform field, and assuming that the energy cascade is isotropic and local
in \( k \)-space, Kolmogorov’s dimensional analysis arguments can be repeated, now with \( \epsilon \) depending on \( k \), \( E(k) \) and \( V_A \) (with dimension \( LT^{-1} \)). Writing \( \epsilon \sim k^\alpha E^\beta V_A^\gamma \), we can deduce the spectral index \( \nu \) of the inertial-range energy spectrum:

\[
\nu = \frac{\alpha + 5 - \gamma}{3 - \gamma}.
\]

(Note that the Kolmogorov spectrum \( \nu = 5/3 \) for HD is obtained as a special case of (2) if we set \( \gamma = 0 \).) To find \( \nu \) for MHD, we must determine \( \nu \). This can be done in the limit of weak turbulence when the lowest-order interaction involves 3-wave interactions during which two wave packets collide for a typical time scale \( \tau_k \sim (kV_A)^{-1} \) and produce a third wave with typical velocity magnitude \( \delta v_1 \sim v_1 \tau \sim v_1^2 / V_A \). By the relation \( \epsilon \sim (\partial v_1)^2 / \tau_k \sim V_A^{-1} \), we obtain \( \nu = -1 \) and thus \( \nu = 3/2 \), which yields the IK spectrum \( E(k) \sim k^{3/2} \).

The scaling results obtained above by dimensional analysis for isotropic MHD turbulence can also be obtained by an alternate heuristic physical argument. Let each of the two colliding wave packets have amplitude of the order \( v_1 \) and spatial scale \( k^{-1} \). We assume again that the energy transfer is local in \( k \)-space, which means that a wave will interact dominantly with another wave characterized by the same length scale but moving in the opposite direction. From the MHD equations, we estimate that \( v_1 \sim k v_1^2 \), where an overdot denotes time derivative. If 3-wave interactions dominate, we can write \( \delta v_1 \sim v_1 \tau \sim v_1^2 / V_A \). In the weak limit, it will take a large number of random collisions, \( N \sim \left( v_1 / \delta v_1 \right)^3 \gg 1 \) to change a wave packet amplitude by a factor of unity. Noting that \( E(k) \sim k^{-1} v_1^2 \), we obtain \( \epsilon \sim v_1^2 / N \tau \sim k^2 E(k) V_A / N \sim k^4 E^2(k) V_A^4 \) which implies again that \( E(k) \sim k^{3/2} \).

The IK theory, which has provided the physical underpinnings of much subsequent work on MHD turbulence, neglects anisotropy. Subsequently, numerous analytical and computational studies have attempted to address different aspects of anisotropic turbulence [32-34,38-43]. In the presence of a uniform magnetic field, the spectrum is anisotropic. Dimensional analysis, by itself, cannot then provide a definite result since it cannot discriminate between the two length scales perpendicular (\( k_\perp^{-1} \)) and parallel (\( k_\parallel^{-1} \)) to the uniform magnetic field. Let us assume that the energy cascade occurs entirely in the direction perpendicular to the uniform field so that the total energy can be written \( \int E(k)dk_\perp dk_\parallel \). (This assumption is shown to be true in §3.) If we now repeat the scaling argument given in the last paragraph with \( \tau_k \sim (k V_A)^{-1} \) and \( E(k) \sim (k V_A)^{-1} v_1^2 \), we obtain

\[
\delta v_1 \sim k_\perp^2 v_1^2 / (k_\parallel V_A) \sim k_\perp^2 E(k) / V_A.
\]

Thus, \( \epsilon \sim k_\perp^4 k_\parallel^2 E^2(k) / V_A^4 \) which implies that the anisotropic spectrum is \( E(k) \sim \epsilon^{1/2} k_\perp^{-2} \) for weak MHD turbulence dominated by 3-wave interactions. In what follows, we test this scaling by a vigorous analytical calculation and numerical simulations.

3. RANDOM THREE-WAVE INTERACTIONS AND THE ANISOTROPIC SPECTRUM

We assume for simplicity that the plasma fluid is permeated by a spatially uniform magnetic field \( \mathbf{B} = \hat{z} \). Then the nonlinear MHD equations can be reduced rigorously to the so-called reduced MHD (RMHD) [44-46] equations

\[
\frac{\partial \Omega}{\partial t} - \frac{\partial J}{\partial z} = [A, J] - [\phi, \Omega],
\]

\[
\frac{\partial A}{\partial t} - \frac{\partial \phi}{\partial z} = -[\phi, A],
\]

where the magnetic field is given by \( \mathbf{B} = \hat{z} + \nabla \times \mathbf{A} \) with \( \mathbf{A} \) as the magnetic flux function, the flow velocity is given by \( \mathbf{v} = \nabla \phi \times \hat{z} \) with \( \phi \) as the stream function, and \( [\phi, A] = \phi A_z - \phi_z A \). Here \( \Omega = -\nabla^2 \phi \) is the parallel vorticity and \( J = -\nabla^2 A \) is the parallel current density. Note that we have normalized the background uniform magnetic field in the \( \hat{z} \)-direction to have unit magnitude, and the density has been chosen so that the Alfvén speed \( V_A = 1 \).

Recent work has demonstrated conclusively that weak MHD turbulence in the presence of a uniform magnetic field is dominated by three-wave interactions.
that mediate the collisions of shear-Alfvén wave packets [31-33]. Using the ideal RMHD equations, Ng and Bhattacharjee (NB) calculate in closed form the three-wave and four-wave interaction terms, and show the former to be asymptotically dominant if the wave packets have non-zero \( k_\perp = 0 \) components. These three-wave interaction terms provide the basis for our Monte-Carlo simulation of the random scattering of Alfvén waves, discussed below.

For weak interactions between two colliding shear-Alfvén wave packets \( f^\pm \) traveling in the \( \pm \hat{z} \) directions, we write perturbative solutions of the form

\[
\phi = f^- (x_\perp, z^-) + f^+ (x_\perp, z^+) + \phi_1 + \phi_2 + \cdots, \tag{6}
\]

\[
A = f^- (x_\perp, z^-) - f^+ (x_\perp, z^+) + A_1 + A_2 + \cdots, \tag{7}
\]

where \( x_\perp = (x, y) \) is perpendicular to \( \hat{z} \) and \( z^\pm = z \mp t \). Here \( f^\pm (x_\perp, z^\pm) \) represents Alfvén wave packets that propagate non-dispersively with the Alfvén speed \( V_A = 1 \). For given zero-order fields \( f^\pm \), we can then calculate the first-order fields from the equations

\[
\frac{\partial \phi_1}{\partial t} - \frac{\partial J_1}{\partial z} = 2 \left[ f^\perp \cdot \nabla f^\perp + [f^\perp, \nabla f^\parallel] \right] = F, \tag{8}
\]

\[
\frac{\partial A_1}{\partial t} - \frac{\partial \phi_1}{\partial z} = 2 \left[ f^\perp, f^\parallel \right] = G. \tag{9}
\]

Equations (8) and (9) are radiation equations for the first-order fields, with the source term determined by the overlap of the given zero-order fields \( f^\pm \) and \( f^\mp \). The asymptotic expressions of \( \phi_1 \), \( A_1 \) can be written,

\[
\phi_1 (x_\perp, t \to \infty) \to f^- (x_\perp, z^-) + f^+ (x_\perp, z^+) \tag{10},
\]

\[
A_1 (x_\perp, t \to \infty) \to f^- (x_\perp, z^-) - f^+ (x_\perp, z^+) \tag{11},
\]

where

\[
f_1^\pm (x_\perp, z) = \pi \int [\tilde{F}(k_\perp, \pm k_\perp, \omega) + \tilde{G}(k_\perp, \pm k_\perp)] e^{ik_\parallel \cdot x} \, dk_{\perp}. \tag{12}
\]

Here \( \tilde{F}(k_\omega, \omega) = \tilde{F}(k_\omega, \omega)/k_{\perp}^2 \) and \( \tilde{F}(k_\omega, \omega) \) is the Fourier transform of \( F(x, t) \), defined by

\[
F(x, t) = \int [\tilde{F}(k_\omega, \omega) e^{ik_\parallel \cdot x} \, dk_{\perp} d\omega. \tag{13}
\]

The Fourier transform \( \tilde{G}(k, \omega) \) is similarly defined. For simplicity, we consider the case where the functions \( f^\pm (x_\perp, z) \) are separable, i.e.,

\[
f^\pm (x_\perp, z) = f^\pm (x_\perp) f^\pm (z) . \tag{14}
\]

NB show that

\[
\tilde{F}(k_\omega, \omega) = \frac{1}{2} \tilde{F}_\perp (k_\omega) \tilde{f}^\perp (\omega), \tilde{F}(k_\omega, \omega) = \frac{1}{2} \tilde{F}_\perp (k_\omega) \tilde{f}^\perp (\omega) \tag{15}
\]

where \( \omega = (k_{\perp} \pm \omega)/2 \), \( \tilde{f}^\perp (\omega) \) is the one-dimensional Fourier transforms of \( f^\perp (z) \), and \( \tilde{F}_\perp \) and \( \tilde{F}_\perp \) are the two-dimensional Fourier transforms of \( F_\perp \) and \( G_\perp \). It follows that

\[
f^\pm (x_\perp, z^\pm) = F_\perp (x_\perp) f^\pm (z^\pm) / 2, \tag{16}
\]

where

\[
u_\perp (x_\perp) = \left[ \int \tilde{F}_\perp (k_\omega) \tilde{F} (k_\omega, \omega) e^{ik_\parallel \cdot x} \, dk_{\perp}. \tag{17}
\]

Imposing periodic boundary condition in \( x_\perp \), we write

\[
f^\pm (x_\perp, t) = \sum \int_{m} f^\pm (x_\perp, t) e^{2\pi im (x_\perp + ny)} \tag{18}
\]

where \( \sum \) are constants. We define the energy

\[
E_\perp (k_{\perp}) dk_{\perp} \text{ with the spectral functions}
\]

\[
E_\perp (k_{\perp}) \propto k_{\perp}^{-\mu_\perp} \text{ or } \int E_{\perp} \propto (m^2 + n^2)^{-\nu_\perp}, \tag{19}
\]

where \( \mu_\perp \) are the spectral indices. Assuming that the energy is randomly distributed in the zeroth-order fields, we can calculate the spectra of the first-order fields using (13). Our main objective is to determine how the spectrum of an Alfvén wave packet changes in time after many collisions with wave packets coming from the opposite direction. To be specific, let
us consider the evolution of a \( f^+ \) field interacting with a sequence of random \( f^- \) fields. We write

\[
\frac{\partial \Psi^+}{\partial t} = -[f^-, \Psi^+] + [f^-, f^+] + [f^- f^+].
\] (17)

where \( \Psi^+ = -\nabla^2 f^+ \). Numerically, the Fourier amplitudes \( f_{mn}^- \) are randomly chosen for a given spectral index \( \mu^- \) in every time step \( \tau \). The time-step is chosen small enough so as to satisfy the weak turbulence assumption and to keep each wave packet in the sequence of \( f^- \) uncorrelated with any other in the sequence. Also, in order to realize a well-resolved inertial range for the \( f^+ \) spectrum, a hyper-dissipation term of the form \( 6 \eta^+ \nabla^2 \Psi^+ \) is added to the right of equation (17). We optimize the simulation so that the inertial range index is insensitive to the value of \( \eta^+ \).

Equation (17) is solved by a pseudo-spectral method for different values of \( \mu^- \) and for different levels of resolution (up to \( 2^{1024} \)) until the \( f^+ \) spectrum reaches a quasi-steady state.

FIGURE 1. The spectra of the \( f^+ \) field with \( \mu^- = 2 \) for different levels of resolution. A vertical separation has been added in between each pair of curves for clarity.

Figure 1 shows the \( f^+ \) spectra for the case with \( \mu^- = 2 \) for different levels of resolution. We see that the inertial range for all runs have roughly the same index, \( \mu^+ \approx 2 \). We have checked numerically the relation \( \mu^+ + \mu^- \approx 4 \) for a range of values of \( \mu^+ \) and \( \mu^- \). For \( \mu^+ = \mu^- = 2 \), which corresponds to the case considered in §2, we obtain the anisotropic energy spectrum \( k^{-2} \), consistent with the heuristic result.

4. CONCLUSION

Prompted by observations of non-Kolmogorov and anisotropic turbulent spectra in the ISM and the solar wind, we have discussed a theoretical model of weak MHD turbulence, which produces an anisotropic energy spectrum proportional to \( k^{-2} \). The anisotropic energy cascade in our model is due to the random scattering of shear-Alfvén waves dominated by three-wave interactions. We have reviewed critically the assumptions underlying the heuristic derivation of scaling laws in HD and MHD turbulence and underscored the need to verify these scaling laws by dynamical calculations. Our dynamical calculation provides independent confirmation of the \( k^{-2} \)-spectrum derived earlier from heuristic arguments [34,35] and wave kinetic theory [33].

Since measurements of fluctuations in the ISM are line-integrated, an interesting question is how the anisotropic spectrum obtained above for a uniform magnetic field might show up in observations of the ISM, which is generally permeated by a spatially varying magnetic field. If we make the drastic but simplifying assumption that the background magnetic field \( B_0 \) takes all possible directions with equal probability, it is easy to show by averaging over three-dimensional wave vector space that the spectrum will be proportional to \( k^{-2} \) [33]. However, because all possible directions for \( B_0 \) are not equally probable, one might expect deviations from the \( k^{-2} \) scaling. This remark is also applicable to observations of anisotropic turbulence in the solar wind which are based so far entirely on single spacecraft observations [48]. Despite this limitation, by analyzing Helios 2 data, Carbone \textit{et al} [27] have obtained significant quantitative information on the three-dimensional structure of anisotropic turbulence in the solar wind.

We conclude with a few cautionary remarks on the limitations of our model. Although one of the strengths of weak turbulence theory is that it provides rigorous closure, it is far from clear that weak turbulence is a valid approximation for ISM or solar wind turbulence, which is often strong and compressive. Furthermore, we have calculated energy spectra, not density fluctuation spectra. It is often assumed that density fluctuations are enslaved to energy fluctuations, but this is not necessarily so [46,49]. In future work, we will attempt to remedy some of these limitations.
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