I. INTRODUCTION

While fast reconnection has been a subject of great interest to laboratory, space, and astrophysical plasma physicists for nearly five decades, the dominant emphasis of nonlinear reconnection theory and simulation has been on understanding steady or quasi-steady reconnection. This is not surprising. The pioneering theoretical models of Sweet–Parker and Petschek, which have provided the compass for organizing our thinking on reconnection physics, are steady-state models. In numerical simulations, there has often been a deliberate attempt to set up initial and boundary conditions that realize a quasi-steady state, thus enabling tests of steady-state models. A recent example is the Global Environment Modeling (GEM) Reconnection Challenge, where reconnection is triggered in a two-dimensional Harris sheet by the imposition of a large perturbation at the neutral line that forces the dynamics to attain a quasi-steady state quickly.

Although such studies are useful in their own right, it should be recognized that steady reconnection is not a generic state, and is frequently never attained in many dynamical situations of great physical interest. In particular, there are phenomena involving magnetic reconnection in laboratory as well as space and astrophysical plasmas where the reconnection dynamics exhibits impulsive behavior, that is, a sudden increase in the time-derivative of the reconnection rate continues to accelerate to produce large magnetic islands that eventually become of the order of the system size, quenching near-explosive growth. By a combination of analysis and simulations, the scaling of the reconnection rate in the nonlinear regime is studied, and its dependence on the electron and the ion skin depth, plasma beta, and system size is determined. © 2005 American Institute of Physics. [DOI: 10.1063/1.1872893]
tude of the electron charge, and $D/Dt = \partial / \partial t + \mathbf{v} \cdot \nabla$ is the total convective derivative. Inclusion of the generalized Ohm’s law, which is essentially a fluid momentum equation for electrons, represents the most fundamental difference between resistive magnetohydrodynamics (MHD) and Hall MHD equations. (In many cases of physical interest, the electron pressure is not a scalar but a tensor. However, the inclusion of a tensor pressure within the realm of a collisionless fluid model introduces issues of closure that call for a kinetic description beyond the scope of the present paper.)

Our focus in this paper is not only on the impulsive nature of reconnection in certain time-dependent physical situations, but also on the specific and important question of whether the reconnection rate in the late nonlinear regime in these situations attains asymptotic values, independent of the mechanism that breaks field lines (that is, resistivity and/or electron inertia), other plasma parameters such as the ion skin depth and plasma beta, and the system size. The scaling of collisionless reconnection has been a subject of controversy in recent years. Guided by the results of the GEM Reconnection Challenge and subsequent work evolving from that exercise, Shay and co-workers’ have made the strong claim that the reconnection rate is not only independent of the mechanism that breaks field lines, but furthermore, that it becomes a “universal constant” of the order of one-tenth of the Alfvén speed, where the Alfvén speed is calculated using the upstream magnetic field strength. While the first part of this claim has been supported by several recent studies (including some of the GEM Challenge studies), the second part of the claim has been questioned by many of these studies. These studies exhibit a broad range of scaling results under different choices of plasma parameters and boundary conditions, and the very existence of this broad range of scaling results, which predict widely different reconnection rates, would appear to refute any claim of universality. However, Shay and co-workers’ have argued that the initial conditions used in most of the dissenting studies produce magnetic island widths of the order of the ion skin depth or less and hence, do not attain the “asymptotic regime” where the “universal scaling” is realized.

In this paper, we revisit the issues mentioned above, using initial conditions for the poloidal field that are very similar to that of Shay and co-workers,” but now in the context of a reduced two-field model of collisionless reconnection valid for low-beta plasmas with a large guide field. (The similarities and differences between the present model and that considered in Refs. 7 and 8 are discussed in more detail in Sec. VI.) This model, which is simpler than the full two-fluid or Hall MHD equations and is amenable to analytical treatment, captures certain essential features of impulsive reconnection dynamics in collisionless low-beta plasmas (such as the tokamak and solar corona). This model involves the electron pressure gradient term in the generalized Ohm’s law which, in the presence of a guide field, has qualitatively similar effects on nonlinear reconnection dynamics as the Hall current term. Previous work has demonstrated nonlinear near-explosive growth of magnetic islands in this model. The same qualitative feature appears in the four-field model, from which the two-field model can be deduced in the low-beta limit by neglecting parallel ion dynamics. Despite its apparent simplicity, the two-field model is computationally challenging, because it involves tracking near-singular and dynamic current sheets in the linear and nonlinear regimes of collisionless reconnection.

The Magnetic Reconnection Code (MRC), recently developed at the Center for Magnetic Reconnection Studies (CMRS) under the auspices of the Scientific Discovery through Advanced Computing (SciDAC) program (supported by the Department of Energy), is an ideal tool for computational studies of the two-field model. To the best of our knowledge, the MRC is the first massively parallel Hall MHD code that incorporates Adaptive Mesh Refinement (AMR). The implementation of AMR enables us to track dynamic near-singular current singularities at very high levels of resolution. By a combination of analysis and numerical computation, we examine the role of near-singular current sheets as drivers of impulsive reconnection, and the asymptotic scaling of the nonlinear reconnection rate in the two-field model.

The following is the layout of this paper. In Sec. II, we introduce the two-field model. In Sec. III, we give comprehensive results on the linear stability of the $m=1$ collisionless tearing mode in the context of the two-field model, making contact with and extending known results in the literature. In Sec. IV, we describe an analytic nonlinear model for the $m=1$ instability. In Sec. V, we give a brief description of the MRC and present numerical results on the nonlinear evolution of the instability. In Sec. VI, we address the important issue of scaling. We conclude in Sec. VII with a discussion of the implications of our results.

II. TWO-FIELD MODEL

The two-field model assumes that the dynamics is two-dimensional, and depends only on the coordinates $x$ and $y$, with $\zeta$ as an ignorable coordinate. The magnetic field is represented as

$$\mathbf{B}(x,y,t) = B_0 \hat{\zeta} + \nabla \phi(x,y,t) \times \hat{\zeta},$$

(3)

where $B_0$ is assumed to be a constant and large guide (or toroidal) field, and $\phi(x,y,t)$ is a flux function. The velocity is represented as

$$\mathbf{v}(x,y,t) = \hat{\zeta} \times \nabla \phi(x,y,t),$$

(4)

where $\phi(x,y,t)$ is a stream function. Detailed derivations of the two-field equations are given in Refs. 19 and 20, and more recently in Ref. 18. The basic physical assumptions are a large toroidal field and low beta, so that the compressional Alfvén wave propogates much faster than any other wave in the system and the fluid motion is essentially incompressible. The two-field equations are given by

$$\frac{\partial F}{\partial t} + [\phi, F] = \rho_s \hat{\zeta} [U, \phi],$$

(5)
\[
\frac{\partial U}{\partial t} + [\phi, U] = [J, \phi],
\]
where \( J = -\nabla^2 \psi \), \( F = \psi + d^2 \), \( U = \nabla^2 \phi \), and the Poisson bracket is defined by \( [\phi, F] = \dot{\psi} \cdot \nabla \phi \times \nabla F \). In Eqs. (5) and (6), all quantities have been made dimensionless. In particular, distance is normalized by the characteristic equilibrium scale \( L_x \) in the x direction, and time is normalized by the poloidal Alfvén time scale \( \tau_A = 4 m_i m_e L_x / B^2_0 \). (Here \( n_0 \) is the equilibrium electron and ion density of a hydrogen plasma, and \( m_i \) is the ion mass.) The two dimensionless parameters are the electron skin depth \( \delta_e = c / (\omega_{pe} L_x) \), where \( \omega_{pe} \) is the electron plasma frequency, and the ion sound gyro radius \( \delta_s = \sqrt{T_e / m_i} L_x = \tau_s / \tau_i \rho_i \) where \( T_e(T_i) \) is the electron (ion) temperature, \( \omega_{ci} \) is the ion cyclotron frequency, and \( \rho_i \) is the ion Larmor radius. The term proportional to \( \rho_i \) on the right of Eq. (5) is due to finite electron compressibility, and can be traced to the electron pressure gradient term in the generalized Ohm’s law (2).}

For the purpose of the present study, periodic boundary conditions are imposed in both \( x \) and \( y \) directions, with the domain of a single periodic cell given by \( -L_x \leq x \leq L_x \), and \( -L_y \leq y \leq L_y \). We choose \( L_x = \pi \) and \( L_y = \pi / \epsilon \), where \( \epsilon = L_y / L_x < 1 \) is the slab aspect ratio. Due to the symmetry properties of Eqs. (5) and (6), we can consider \( \psi(\phi) \) to be an even (odd) function in both \( x \) and \( y \) for all time, if it is so initially. In what follows, we consider the linear and nonlinear evolution of the equilibrium given by \( J_0 = \psi_0 = \cos x \), \( U_0 = \phi_0 = 0 \). This equilibrium has also been studied in considerable detail in Refs. 8, 20, 23, and 24. It is doubly periodic, and is unstable with respect to double tearing modes which grow around resonant surfaces located at \( x=0 \) and \( \pm \pi \) in the periodic cell.

**III. LINEAR INSTABILITY**

To determine the linear instability of equilibria that depend only on \( x \), we write \( \psi = \psi_0 + \psi_1(x,t) \cos ky \), \( \phi = \phi_0 + \phi_1(x,t) \sin ky \), where \( k = \epsilon m \) and \( m \) is an integer. For fixed \( m \), the wave number \( k \) is linearly proportional to the aspect ratio \( \epsilon \). Linearizing Eqs. (5) and (6), we obtain

\[
\frac{\partial}{\partial t} \left[ (1 + k^2 d^2) \psi_1 - d^2 \frac{\partial^2 \psi_1}{\partial x^2} \right] = k \sin x \left[ \rho_e^2 \frac{\partial^2 \phi_1}{\partial x^2} - (1 + d^2) \frac{\partial^2 \phi_1}{\partial x^2} \right],
\]

\[
= k \sin x \left[ (1 - k^2) \psi_1 + \frac{\partial^2 \phi_1}{\partial x^2} \right].
\]

The linear dispersion relation and growth rate of collisionless tearing modes in the two-field model have been obtained analytically using boundary-layer and asymptotic matching techniques. The analytic theory is mostly based on the large-\( \Delta' \) approximation, i.e.,

\[
\Delta' \delta_x \sim \min[1, (d \rho_i)^{1/3}]
\]

where \( \Delta' = 2 \sigma \tan(\sigma \pi / 2) \), \( \sigma = \sqrt{1 - k^2} \). The parameter \( \Delta' \) is positive for \( 0 < k \leq 1 \), which is necessary for instability. The large \( \Delta' \) regime generally requires small \( k \) because \( \Delta' \) is proportional to \( k^{-2} \) for small \( k \), that is, large \( \epsilon \). When \( 0.5 < \epsilon < 1 \), only the \( m=1 \) mode is linearly unstable. For \( \epsilon < 0.5 \), a larger range of \( m \)-numbers are destabilized, up to a maximum mode number equal to integer \( (\epsilon^{-1}) \).

The growth rate can be calculated by the initial-value method numerically, whereby the equilibrium is perturbed by noise and Eqs. (7) and (8) are integrated numerically until the solutions develop into the eigenmode with the largest growth rate. In this section, we describe yet another direct and efficient way to calculate the growth rate numerically which will be compared with results obtained by the initial-value method and with known analytic results. Our method, which is similar to a method developed recently to study the complete kinetic spectrum of a weakly collisional plasma, can provide comprehensive results on instability growth rates for arbitrary values of the parameters \( k, d \), and \( \rho_i \).

Exploiting the symmetry properties of the solutions mentioned in Sec. II, we write

\[
\psi_1(x,t) = \sum_{n=0}^{\infty} a_n(t) \cos nx, \quad \phi_1(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin nx.
\]

Substituting Eq. (10) into Eqs. (7) and (8), we obtain two recurrence relations. The first one is

\[
2[1 + d^2(k^2 + n^2)]a_n / k = [1 + d^2 + k^2 \rho_i^2 + (n-1)^2 \rho_i^2]b_{n-1} - [1 + d^2 + k^2 \rho_i^2 + (n + 1)^2 \rho_i^2]b_{n+1},
\]

where overdot denotes time derivative, and \( b_0 = b_{-1} = 0 \). The second relation is

\[
2(k^2 + n^2)b_n / k = [k^2 - 1 + (n - 1)^2]a_{n-1} - [k^2 - 1 + (n + 1)^2]a_{n+1}
\]

for \( n > 1 \), and

\[
2(k^2 + 1)b_0 / k = [k^2 - 1]a_0 - [k^2 + 3]a_2
\]

for \( n = 1 \). By combining Eqs. (11), (12a), and (12b), we can relate the amplitude \( a_n \) with the amplitudes \( a_{n+2} \) and \( a_{n-2} \) for a given eigenfrequency \( \omega \). The amplitude \( a_n \) obeys the equation \( \dot{a}_n = -\omega^2 a_n \), where negative \( \omega^2 \) produces instability. It can be shown that the amplitude \( a_n \) must have either \( n \) all even or all odd. Our numerical results, discussed below, suggest that for all unstable modes, \( n \) is all even.

Our numerical strategy for calculating the eigenvalues is as follows. We note that by simply using Eqs. (11), (12a), and (12b) and iterating forward from \( n=0 \) to \( n \rightarrow \infty \) for given \( \omega \) will not produce a converged series for \( a_n \) even if \( \omega \) is an eigenvalue. To see this, let us consider the limit \( n \rightarrow \infty \). The recurrence relations (11), (12a), and (12b) yield

\[
4d^2 \omega^2 a_n = \rho_i^2 k^2 [a_{n+2} - 2a_n + a_{n-2}].
\]

If \( a_{n+2} / a_n \rightarrow r \) as \( n \rightarrow \infty \), we obtain, for an unstable mode \( (\omega^2 < 0) \),
where the approximation holds if $d_e/\omega/\rho_1 k$ is much smaller than unity. It follows that the series $a_n$ can either diverge ("large" solution) or converge to zero ("small" solution) as $n \to \infty$, where the latter behavior corresponds to a physically acceptable eigenfunction. Thus, the eigenvalue problem for a physically acceptable eigenfunction reduces to a search for $\omega^2$ so that the condition $a_n \to 0$ as $n \to \infty$ is met. However, iterating the series forward generally tends to produce the divergent series due to numerical roundoff and instability. To ensure that we obtain the convergent series during iteration, we start out with a large value of $n$ and iterate the recurrence relations backward. This ensures that the backward series stays close to the small solution, but the backward series will not match the forward series unless $\omega^2$ is an eigenvalue. We determine the eigenvalue numerically by iterating the two series until they match in their prediction for $a_2/a_0$ by a minimization procedure (such as the downhill simplex method) to a specified level of accuracy.

We have compared the results from the method based on recurrence relations, described above, with those from the initial-value method implemented in the MRC and they are seen to be in good agreement. This gives us a way to benchmark the MRC, which is used to study the nonlinear evolution of linearly unstable collisionless tearing modes in the two-field model. Our results generally confirm the predictions of analytic linear theory\textsuperscript{20,23-25} when the parameters $d_e$ and $\rho_1$ are in the asymptotic regime where the theory applies. As mentioned above, we are interested here primarily in the regime of large $\Delta'$, which generally requires small $k$. However, as shown in Fig. 1, for the case $\rho_1=0.1$, $d_e=0.01$, the linear growth rate $\gamma = \sqrt{-\omega^2}$ is a strong, nonmonotonic function of $k$ and usually peaks at a value of $k$ that is not very small. If we require that a single $m=1$ mode be unstable, the aspect ratio must have a lower bound given by $\varepsilon=0.5$, which yields an upper bound on $\Delta'$ given by $\Delta' \approx 8.1$. This then restricts the range of physical parameters over which the large-$\Delta'$ approximation is valid for a single mode.

In the case $d_e \gg \rho_1$, when $\Delta'$ is large in the sense of inequality (9), the analytic theory predicts the linear growth rate $\gamma_L = kd_e$. In the small-$\Delta'$ regime, that is, when the inequality (9) is not satisfied, the linear growth rate becomes $\gamma_L = kd_e$. Figure 2(a) shows that our numerical results agree with these theoretical predictions in this case with $\rho_1=0$, $k = 0.5$. However, we note that in this case, for most values of $d_e$ of physical interest, the linear mode is in the small-$\Delta'$ regime and makes a transition to the large-$\Delta'$ regime when $d_e$ is of the order unity. Figure 2(b) shows the same curves for $k=0.01$, which corresponds to $\Delta' \approx 25463$. Now one sees the growth rate for the fastest growing mode conform to the theoretical prediction $\gamma_L = kd_e$ over a much larger and physically interesting range of $d_e$.

For the case $d_e \ll \rho_1$, which is of interest for tokamaks (as well as the solar corona), the analytic theory predicts $\gamma_L = k(d_e/\rho_1)^{1/3}$ in the large-$\Delta'$ regime. Figure 3(a) shows the $\gamma_L \propto d_e^{1/3}$ scaling for the case $\rho_1=1$, while Fig. 3(b) shows the $\gamma_L \propto \rho_1^{1/2}$ scaling for the case with $d_e=0.001$. Overall, although the linear analytic theory gives fairly accurate growth rates in its domain of validity, the general functional dependence of the growth rate on $k$, $d_e$, and $\rho_1$ is more delicate outside of the range of validity of the analytic theory, and needs to be determined numerically.

**IV. NONLINEAR EVOLUTION: AN ANALYTICAL MODEL**

In the case $d_e \neq 0$, $\rho_1=0$, Ottaviani and Porcelli\textsuperscript{23,24} (hereafter, OP) have derived an island equation analytically using the conservation of $F=\psi+J^2$, which is an exact invariant convected by the fluid when $\rho_1=0$. (In the absence of resistivity, the reconnection process is, in principle, revers-
ible in time.) The equation derived by OP, which describes the time evolution of the island half-width $w$, defined by the relation $\phi(0,0,t) = 1 - w^2(t)/2$, is given by

$$\frac{d^2 \hat{w}}{dt^2} = \frac{1}{4} (\hat{w} + C \hat{w}^4),$$  \hspace{1cm} (15)$$

where $\hat{t} = \gamma_t t$ is the time variable normalized by the linear growth rate $\gamma_t$, $\hat{w} = w/d_e$, and $C$ is a positive quantity, slowly varying in time, of the order of 1/4. We note that $\phi(0,0,t=0) = 1$ and $w(0) = 0$, and we have included a factor of 1/4 in the first term of Eq. (15) because $w$ grows exponentially with half of the linear growth rate $\gamma_t$. By the time $w$ becomes of the order of the system size, most of the magnetic flux has been reconnected. Equation (15) has the property that if $\gamma_t \to 0$, which occurs when $d_e \to 0$, the linear as well as the nonlinear reconnection rate tends to zero.

The applicability of Eq. (15) when $\rho_s \neq 0$ is open to question because the method used by OP to conservation of the quantity $F = \phi + d_e^2 J$, which is not conserved when $\rho_s \neq 0$. Here we extend the results obtained by OP to the case $\rho_s \neq 0$ by an independent method that does not rely on the conservation of $F$. We focus here on the origin ($x = 0$), which is the location of an X point in the $x$--$y$ plane. From Eqs. (5) and (6) and the symmetry properties of the equilibrium and the dynamical relations, we have the following exact relations at $x = 0$:

$$\dot{U}_{xy} = (F_{xx} \psi_{yy} - F_{yy} \psi_{xx})/d_e^2,$$  \hspace{1cm} (16a)$$

$$\dot{F}_{xx} = 2(F_{xx} \psi_{yy} - \rho_s^2 U_{xy} \psi_{xx}),$$  \hspace{1cm} (16b)$$

$$\dot{F}_{yy} = 2( -F_{yy} \psi_{xx} + \rho_s^2 U_{xx} \psi_{yy}),$$  \hspace{1cm} (16c)$$

where the subscripts $x$ or $y$ stand for partial derivative with respect to $x$ or $y$, evaluated at the origin. These equations of $U_{xy}$, $F_{xx}$, and $F_{yy}$ do not form a closed set since they depend on unknown quantities $\phi_{xx}$, $\phi_{xy}$, and $\phi_{yy}$. In what follows, we assume that a single $m = 1$ mode (usually the fastest growing mode) dominates in the linear as well as the nonlinear regime. However, when the dynamical evolution leads to the growth of a large spectrum of $m$ modes, either because there is a large spectrum of linearly unstable modes or more importantly, because nonlinear evolution involves a larger spectrum due to mode coupling, the specific ansatz underlying the analytical model becomes questionable. Despite this caveat, the nonlinear model appears to capture the near-explosive growth of the $m = 1$ magnetic island seen in the nonlinear stage of the simulations for a wide range of parameters.

The specific ansatz we use is based on numerical evidence, and is similar to the one used by OP. We observe that the spatial structure of the stream function $\phi(x,y,t)$ for short times (when linear theory is valid) is similar to that for long times when the system evolves into the nonlinear phase. This observation motivates us to assume the following form of $\phi_{xy}$ and $U_{xy}$ at $x = 0$ in the nonlinear regime:

$$\phi_{xy} \approx \hat{w} / \delta_t,$$  \hspace{1cm} (17a)$$

$$U_{xy} \approx - \hat{w} / \delta_t \delta_x^2.$$  \hspace{1cm} (17b)$$

Here $\delta_t$ represent the linear boundary layer width, proportional to $d_e$ when $\rho_s = 0$, or $d_e^{2/3} \rho_s^{1/3}$ when $\rho_s \neq 0$ and $\rho_s \approx d_e$. It continues to represent approximately the spatial scale of the velocity even in the nonlinear regime. Equation (17b) may be viewed as a defining relation for the spatial scale $\delta_t$ along $x$. The scale $\delta_t$ is of the order of $d_e$ in the linear phase, and tends to decrease further in the nonlinear phase. In Eq. (17), $\hat{w}$ represents $u$, the incoming flow velocity in the $x$ direction toward the origin.

With the ansatz (15), we can make some scaling estimates. Using $\hat{F} = 0$ at $x = 0$, we can write

$$F = 1 + d_e^2 = 1 - w^2/2 + d_e^2 J.$$  \hspace{1cm} (18)$$

From Eq. (18), we obtain $J = -\psi_{xy} - \psi_{yy} = 1 + w^2/2d_e^2$, where we assume that $d_e^2 \ll 1$. Estimating $\psi_{xy} = k^2 w^2/2$, we obtain from Eq. (18), $\psi_{xy} = -w^2/2d_e^2 + k^2 w^2/2$. Furthermore, we write $J_{xx} = -c_j w^2/2d_e^2 \delta_t^2$, $J_{yy} = -k^2 w^2/2d_e^2$, $F_{xx} = -c_j w^2/2d_e^2 \delta_t^2$, and $F_{yy} \approx 0$, where $c_j$ is a constant of the order of unity. Substituting these estimates into Eq. (16a), we obtain

$$- \dot{U}_{xy} = d_e^2 \frac{\hat{w}}{\delta_t \delta_x^2} \approx - F_{xx} \psi_{xy} / d_e^2 \approx \frac{c_j k^2 w^4}{4d_e^2 \delta_t^2}. \hspace{1cm} (19)$$

We can also estimate the time evolution of $\delta_t$ in the nonlinear phase by using Eq. (16b). Substituting the estimates given above into Eq. (16b), assuming that $\delta_t$ changes with time while $\delta_t$ remains a constant and that $w$ is close to unity in the nonlinear phase, we obtain

$$\dot{\delta}_t = - \frac{w \delta}{\delta_t} \left[ 1 + \frac{\rho_s^2}{c_j \rho_s^2} \right], \hspace{1cm} (20)$$

which can be integrated to yield

$$\delta_t \approx \delta_t \exp \left[ - \frac{w}{\delta_t} \left( 1 + \frac{\rho_s^2}{c_j \rho_s^2} \right) \right]. \hspace{1cm} (21)$$

Since $w$ exhibits very rapid growth, the spatial scale $\delta_t$ decays super-exponentially. In fact, Eqs. (19) and (21) predict that in the nonlinear regime the current channel width tends to zero and the island width tends to blow up in finite time. We caution that this is not a true finite-time singularity for two reasons: first, Eq. (19) itself breaks down when the island becomes of the order of the system size and second, dissipation (such as resistivity or hyper-resistivity, either physical or numerical), no matter how small, is a regularizing influence. In the numerical results presented below, the nonlinear decay of $\delta_t$ to zero is thwarted by resistivity. If we make the assumption that $\delta_t$ and $\delta_t$ tend to constant values due to the intervention of resistivity, add a term proportional to $w$ on the right of Eq. (19) to describe the linear phase, normalize time by the linear growth rate $\gamma_t$ (equal to $kd_e^{2/3} \rho_s^{1/3}$) and spatial scales by the linear boundary-layer width $\delta_t$ (equal to $d_e^{2/3} \rho_s^{1/3}$), we recover a nonlinear island equation of the form (15), that is,
\[
\frac{d^2 \hat{w}}{dt^2} \approx \frac{1}{4} (\hat{w} + c_f \hat{w}^4),
\]

where \( \hat{w} = w/\delta_x, \) \( t = \gamma_t t \) and \( c_f = 1/4 \) is a constant. The choice \( c_f = 1/4 \) reproduces Eq. (15), the equation derived by OP.

It is worth noting that the tendency for the formation of a current singularity and island blow-up is already inherent in this problem due to the presence of finite electron inertia even when \( \rho_e = 0 \). When \( \rho_e \neq 0 \), the tendency for current singularity formation persists, although the characteristic spatial and temporal scales (such as \( \delta_x, \gamma_t, \) and \( \delta ) \) then depend on \( \delta_x \) as well as \( \rho_e \). It is in this sense that we describe the current singularity as a driver of impulsive reconnection. The analytical model also predicts that the linear as well as the nonlinear reconnection rate does depend on the system size (that is, \( k \)). (As Fig. 1 shows, the linear growth rate has a strong dependence on \( k \), which is a linear dependence in the large-\( \Delta' \) regime.) In the following section, we will test the predictions of this analytical model with numerical simulations.

V. SIMULATION RESULTS FROM THE MAGNETIC RECONNECTION CODE

The Magnetic Reconnection Code (MRC) is one of the principal computing tools developed at CMRS in the last two years. The MRC integrates the Hall MHD or two-fluid equations in a parallel AMR framework. As discussed above, fast reconnection in the two-fluid or Hall MHD framework is often characterized by the development of thin current sheets, evolving rapidly in space and time. AMR is an ideal computational tool for such problems. Starting out from a uniform grid, the local truncation errors are monitored and new, and refined grids, referred to as higher levels of AMR, are introduced locally as needed. In addition, buffer cells are included in the direction of the anticipated movement of the small-scale structures to ensure sufficient resolution until the next refinement step occurs. AMR is very efficient for problems where localized small scales occur over small volumes (typically not exceeding about one-third of the total computing volume). It enables us to follow the evolving near-singular structures with a precision that even the most powerful supercomputer cannot attain when the simulation is run on a uniform grid. The MRC applies to plasmas as well as fluids (which obey the Navier–Stokes equation). The code is set up to use a variety of space and time integration schemes. For the results presented we have used a Runge–Kutta time integrator combined with simple finite differences as well as a much more sophisticated Central Weighted Essentially Non-oscillatory scheme.\(^{27-30}\) A major challenge for the numerical integration of the Hall MHD (or Navier–Stokes) equations is the need to solve elliptic equations on the grid hierarchy generated by the AMR framework. Our underlying strategy is a variant of the fast adaptive composite algorithm developed by McCormick.\(^{31}\) We integrate this algorithm within the framework of the Portable Extensible Toolkit for Scientific Computation (PETSc), giving us easy access to a large variety of preconditioners and Krylov-accelerators. At the same time, we can also exploit PETSC’s optimized computational kernels and integrated parallelization support. The MRC has been run on a hierarchy of fully refined grids and shown to scale satisfactorily up to the highest used processor count of 1024 on the IBM SP machine, Seaborg. More details on the MRC have been given by Germaschewski et al.\(^{32}\)

In Fig. 4, we compare the linear growth rates calculated by the MRC (indicated by diamonds) for the \( m=1 \) collisionless tearing mode with the growth rates obtained by the recurrence relation approach discussed in Sec. III. The quality of the agreement between the two approaches gives us confidence in the accuracy of the MRC.

Figure 5 is a typical image plot of the current density \( J(x,y,t) = -\nabla_y^2 \psi \) in the nonlinear regime. This picture illustrates the usefulness of AMR grids in resolving intense and thin current sheets produced during collisionless reconnection dynamics. The magnified images in the smaller inserts show clearly the detailed spatial structure in the vicinity of an \( X \) point, and the AMR grids used to resolve them.

In Figs. 6(a) and 6(b), we plot image plots of the flux function \( \psi(x,y,t) \) and the current density \( J(x,y,t) = -\nabla_y^2 \psi \), respectively, at different instants of time for the parameters \( d_e = 0.1, \rho_e = 0.2, \) and \( \epsilon = 0.5 \) during the evolution of the reconnection dynamics, including the early and late nonlinear phases. Even though we have chosen a relatively large value of \( d_e/\rho_e \) in this example to enable better visualization of the near-singular dynamics, the current sheet thinning occurs very rapidly and slows down eventually due to the intervention of resistivity (either physical or numerical), while the island grows near-explosively in time. As the island size becomes of the order of the system size, the near-explosive growth is quenched.

Figure 7(a) shows a plot of \( J^{-1}(\partial^2 J/\partial x^2) \) at the origin, which is essentially the reciprocal of the square of the current sheet width. After a period of exponential growth, this quantity tends to blow up very rapidly, approximately consistent with the analytical prediction, given by Eq. (21). We show this plot for three different levels of AMR. It is clear by inspection that the higher the level of AMR, the longer the blow-up phase, before the process saturates. Figure 7(b) shows a plot of the time evolution of the island width.

In Figs. 8(a), 8(b), and 8(c), we plot, respectively, cross sections of \( \phi(x,y,t), J(x,y,t), \) and \( U(x,y,t) \) as a function of
$x$ at fixed values of $y$ in the linear as well as nonlinear regimes. Figure 8(a) supports the ansatz that the spatial structure of the stream function $\psi(x,y,t)$ for small times (when linear theory is valid) is similar to that for long times when the system evolves into the nonlinear phase. On the other hand, Figs. 8(b) and 8(c) show clearly that the spatial scales of $J(x,y,t)$ and $U(x,y,t)$ in the reconnecting layer, represented by the shrinking scale $\delta_x$, shrink rapidly as the system evolves from the linear to the nonlinear phase.

In Fig. 9, we compare the simulation result from the MRC with the island equation (22) for the same parameters as Fig. 8. (Note that the ordinate is plotted on a logarithmic scale.) As mentioned in Sec. IV, the constant $c_J$ is not fixed by our analysis. We find that $c_J \approx 0.1$ provides a reasonably good fit for the simulation results in this case. We caution that the quality of the fit should be taken with a grain of salt.
because the value of the constant \( c_J \) chosen to provide the fit is not universal for all parameters and varies over a range of order unity for different choices of parameters.

VI. IS THE ASYMPTOTIC RECONNECTION RATE A UNIVERSAL CONSTANT?

In two papers, Shay and co-workers have presented extensive numerical results in support of their claim that the reconnection eventually evolves into a late nonlinear phase, which they call the “asymptotic phase,” when the reconnection rate becomes of the order of one-tenth of the Alfvén speed (based on the magnetic field just upstream of the reconnection layer), independent of the electron and the ion skin depth as well as the system size. Although counterexamples to this claim have been discussed in Refs. 10, 13, and 18, Shay et al. have argued recently that none of the counterexamples go far enough for two reasons: first, these studies measure the reconnection rate when the island widths are of the order of \( d_i \) (or \( \rho_s \)) which is not large enough to realize a true asymptotic phase, and second, the reconnection rate is not measured in terms of the magnetic field upstream of the dissipation region.

We revisit this question here, especially because the initial condition for the poloidal field used by Shay and co-workers is very similar to the one used here. There are, however, significant differences between our model and theirs. Shay and co-workers use the full Hall MHD equations and take the equilibrium guide field to be zero. We consider an equilibrium with a large and constant guide field, and integrate the reduced two-field equations which are obtained
from the full Hall MHD equations in the limit of large toroidal field and low beta. Despite these differences, it is instructive to compare our results with those of Shay et al.,\textsuperscript{8} because there is no doubt that both simulations exhibit an "asymptotic phase."

Shay et al.\textsuperscript{8} make the reasonable but somewhat ad hoc suggestion that the upstream edge of the dissipation region be determined by taking the location where the ion current is 25\% of its maximum value. While we are unable to implement this suggestion in the context of the two-field model (because the parallel current is carried entirely by electrons in this model), we suggest that the instantaneous reconnection rate, as measured by the rate of change of the island width, which is proportional to the inflow velocity towards the X-point, is a good diagnostic with which we can test the claim regarding asymptotic reconnection rates. One of the advantages of the two-field model is that it provides an analytical model for the nonlinear evolution of islands, which may provide insights on scaling dependencies that may be difficult to determine entirely from the results of numerical simulations.

Figure 10 shows five plots of the island width as a function of time for the parameters $d_e=0.25$, and $\rho_s=0, 0.25, 0.5, 0.75, 1.0$, holding the aspect ratio $\alpha$ fixed at the value 0.5. Figure 10 shows the same five plots for the island width, now as a function of $\gamma_L t$, where $\gamma_L$ is the linear growth rate, determined numerically from the MRC. We observe that the five plots essentially lie on top of each other for most of the time interval during the evolution of the instability. In other words, the equation $w=w(\gamma_L t)$ is a reasonably good description for the island width evolution for most of the time interval.\textsuperscript{20} Had this been the whole story, the issue of scaling of the reconnection rate would be completely settled, and we could claim, following Refs. 15 and 20, that the instantaneous reconnection rate, as measured by the rate of change of the island width (which is proportional to the inflow velocity towards the X-point), is a good diagnostic with which we can test the claim regarding asymptotic reconnection rates. One of the advantages of the two-field model is that it provides an analytical model for the nonlinear evolution of islands, which may provide insights on scaling dependencies that may be difficult to determine entirely from the results of numerical simulations.

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We note, however, that the equation $w=w(\gamma_L t)$ is not quite the whole story. The five curves in Fig. 10(b) do not lie on top of each other in the late nonlinear phase, although they tend to be quite close to doing so as the ratio $d_e/\rho_s$ becomes larger. [For the numerical results presented by
Grasso et al.,\textsuperscript{20} these deviations from the equation $w = w(\gamma_L t)$ in the late nonlinear phase are not as obvious from their Fig. 7, which gives a plot of $w(t)/d_e$ vs $\ln(\gamma_L t)$ and thus appears to compress the differences seen in Fig. 10(b).\textsuperscript{19} In order to determine numerically the scaling behavior of the time-dependent growth rate in the late nonlinear (or asymptotic) phase, we choose to examine the nonlinear growth rate at a fixed size of the island width ($w=2$), which falls right in the middle of the late nonlinear phase. (The choice $w=2$ is admittedly ad hoc, but our qualitative conclusions regarding the late nonlinear phase do not depend on this specific choice.) Figure 11 shows the plots of $\gamma_{NL}$ for the five values of $\rho$, given above at fixed island size. For comparison, we also plot $\gamma_L$ for the same values of $\rho$. Table I shows the ratio of $\gamma_{NL}/\gamma_L$ in this case; the ratio varies from approximately 1.5 to 2.3. From inspection of Fig. 11 and Table I, we conclude that although the growth rate in the late nonlinear regime shows deviation from the equation $w = w(\gamma_L t)$, this growth rate scales with $\rho$, in approximately the same way as $\gamma_L$ does. A similar conclusion holds for the dependence of the growth rate on $d_e$ in the late nonlinear phase.

We now investigate the dependency of the asymptotic growth rate on the aspect ratio $\varepsilon$. Figure 12 shows five plots of the island width as a function of $\gamma_L$, where $\gamma_L$ is the linear growth rate, determined numerically for fixed $d_e=0.25$ and $\rho=0.75$, and five different values of the aspect ratio given by $\varepsilon=0.1, 0.2, 0.3, 0.4, 0.5$. We note that for $\varepsilon<0.5$, more than a single mode is linearly unstable, but the nonlinear evolution is dominated by the fastest growing mode. Once again, we observe that the five plots essentially lie on top of each other for most of the time interval during the evolution of the instability. In other words, in this case too the equation $w = w(\gamma_L t)$ is a reasonably good description for the island width evolution for most of the time interval.\textsuperscript{20} This would imply a strong dependence of the reconnection rate on $k$, in contrast with the conclusion of Ref. 8. However, as in Fig. 10(b), we note that the five curves in Fig. 12 do not lie on top of each other in the late nonlinear phase. So in order to determine numerically the dependence of the time-dependent growth rate on the aspect ratio in the late nonlinear phase, we choose to examine the nonlinear growth rate at a fixed size of the island width ($w=2$). Figure 13 shows the plots of $\gamma_{NL}$ for different values of $\varepsilon$ at fixed island size. For comparison, we also plot $\gamma_L$ for the same values of $\varepsilon$. Table II shows the ratio of $\gamma_{NL}/\gamma_L$ in this case; the ratio varies from approximately 1.5 to 2. From inspection of Fig. 13 and Table II, we conclude that although the growth rate in the late nonlinear regime shows deviation from the equation $w = w(\gamma_L t)$, this growth rate clearly exhibits a definite dependency on $\varepsilon$. However, this dependency of the nonlinear growth rate on $\varepsilon$ appears to be a little weaker than the dependency of $\gamma_L$ on $\varepsilon$.

### VII. SUMMARY

In this paper, we have presented linear and nonlinear results on collisionless reconnection in a two-field model, valid in the regime of high guide (or toroidal) field and low plasma beta. In this model, electron inertia breaks field lines, and two-fluid (or Hall MHD) effects enter via the electron pressure gradient term in the generalized Ohm’s law. The two parameters representing electron inertia and pressure

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$\gamma_{NL}/\gamma_L$</th>
</tr>
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<tbody>
<tr>
<td>0.2</td>
<td>1.96</td>
</tr>
<tr>
<td>0.3</td>
<td>1.64</td>
</tr>
<tr>
<td>0.4</td>
<td>1.54</td>
</tr>
<tr>
<td>0.5</td>
<td>1.56</td>
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</table>

![FIG. 12. Island width time evolution for different values of $\varepsilon$ for $\rho_e=0.75$, $d_e=0.25$ in rescaled (linear growth rate) time.](image)

![FIG. 13. Linear and nonlinear growth rates (at fixed island size) for the runs in Fig. 12.](image)
gradient are $d_s$ and $\rho_s$, respectively. The initial condition chosen is spontaneously unstable to the $m=1$ tearing instability. Even if $\rho_s=0$, the system of equations exhibit near-explosive nonlinear growth of current sheet amplitude and magnetic island width. In the regime $\rho_s \neq 0$, $\rho_s > d_s$, the tendency for near-explosive growth persists, but we repeat for emphasis that this tendency is already inherent in the system without $\rho_s$. Thus, in the present model, current singularities drive impulsive reconnection, and it is not surprising that the scaling properties of this system exhibit dependency not only on $\rho_s$ but also on $d_s$, which breaks field lines and controls the structure of the current sheet.

Despite its apparent simplicity, the two-field model is computationally challenging, and we have presented here the first results from the MRC, which integrates the Hall MHD equations in an AMR framework. We have benchmarked the MRC with an independent method, based on recurrence relations, that calculates the linear growth rate of the $m=1$ instability accurately and efficiently for arbitrary values of $d_s$, $\rho_s$, and the aspect ratio $\epsilon$. In the nonlinear regime, the AMR capabilities enable us to resolve the near-singular growth of current sheets and magnetic islands until the growth is quenched due to the intervention of resistivity (either numerical or physical) and large island-size (of the order of the system-size). We have proposed an analytical model, extending earlier work by Ottaviani and Porcelli,\textsuperscript{23,24} that accounts for the near-explosive dynamics seen in the simulations. This type of dynamics is similar to that studied in great detail by Shay and co-workers,\textsuperscript{8} who have suggested that the reconnection rate tends to a “universal” rate of the order of one-tenth of the Alfvén speed (where the Alfvén speed is calculated using the upstream magnetic field strength) in the late nonlinear regime. We have demonstrated that the reconnection rate in the late nonlinear regime of the two-field model attains no such “universal” behavior, but depends on $d_s$ and $\rho_s$ in approximately the same way as the linear growth rate. We have also demonstrated that this reconnection rate depends on the aspect ratio (or the system size), although this dependency is a little weaker in the late nonlinear regime than it is in linear theory.

As discussed above, one of the significant qualitative consequences of the present study is that the dynamics and scaling properties of Hall MHD or two-fluid collisionless reconnection models is not independent of the mechanism that breaks field lines. In the present context, electron inertia is that mechanism, and it introduces filamentary and rapidly time-varying current density structures that persist through the linear as well as nonlinear regimes, and produce dynamics that is quite different than resistive MHD dynamics. That this is so for linear theory has been known for a long time, but the effect persists also in the nonlinear regime of the present model. Thus, in problems of time-dependent collisionless reconnection, current singularities that are dominantly controlled by electron inertia cannot, in general, be assumed to be a sideshow to ion-controlled reconnection. The nature of the current sheet dynamics is qualitatively different if electron inertia replaces resistivity as the mechanism that breaks field lines, and this can produce significantly different scaling dependencies. Such differences in current sheet dynamics play an important role in nonlinear forced reconnection studies as well, as can be readily seen by comparing the simulation results reported in Refs. 33 and 34.

ACKNOWLEDGMENTS

This research is supported by the Department of Energy under the auspices of the program on Scientific Discovery through Advanced Computing, and the National Science Foundation. A.B. acknowledges gratefully the hospitality of the staff and stimulating discussions with visiting members of the Isaac Newton Institute for Mathematical Sciences at the University of Cambridge, where part of this work was completed.