Four-field model for dispersive field-line resonances: Effects of coupling between shear-Alfvén and slow modes

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Abstract. A new theoretical model is proposed for dispersive field-line resonances in collisionless magnetospheric plasmas on the basis of reduced four-field equations. The model improves upon the predictive capabilities of earlier two-field models. In particular, due to the coupling of the shear-Alfvén mode to the slow mode in the four-field system, it is now possible to account for the observed low frequencies of field-line resonances. Furthermore, parallel electric fields can be large without requiring the field-aligned current density to be unrealistically large. Qualitative implications for recent FAST and ground-based observations are discussed.

Introduction

A field-line resonance (hereafter, abbreviated as FLR) describes the natural mode of oscillation of a near-Earth geomagnetic field line with footpoints anchored firmly in the highly conducting northern and southern ionospheres. In an ideal plasma, this mode of oscillation can be identified, under certain conditions, as a shear Alfvén wave which obeys the dispersion relation

\[ \omega^2 = k^2 || V_A^2 \]

where \( \omega \) is the eigenfrequency, \( k || \) is the component of the wavenumber \( k \) parallel to the equilibrium magnetic field, and \( V_A \) is the Alfvén speed. The ideal shear-Alfvén wave cannot support a parallel electric field. Hasegawa [1976], and subsequently Goertz and Boswell [1979], pointed out that if nonideal effects (electron inertia, electron pressure and/or finite ion Larmor radius) are included in the theory, one obtains the kinetic or inertial Alfvén wave, which can support a parallel electric field. Such a parallel electric field can accelerate electrons [Hasegawa, 1976; Haerendel, 1983; Lysak and Dum, 1983; Goertz, 1984; Hui and Sckyler, 1992; Kletzing, 1994; Wei et al., 1994] and provide a potentially promising mechanism linking FLRs and small-scale auroral structures including discrete auroral arcs [Xu et al., 1993; Samson et al., 1996].

In a sequence of recent papers, Streltsov and Lotko [Streltsov and Lotko, 1996, 1997] have developed a linear model of FLRs that is quite successful in accounting for several features of an event observed by the FAST satellite. The Streltsov Lotko model, when extended to include gyrokinetic effects [Streltsov et al., 1998] and an anomalous parallel resistivity, reproduces the local electric and magnetic field signatures as well as the downward electron energy flux observed by FAST in striking detail [Lotko et al., 1998].

Despite these accomplishments, Lotko et al. [1998] have identified a few significant issues that remain unresolved. The typical oscillation frequency of the FLR predicted by the model is larger by nearly an order of magnitude compared with that inferred from ground-based observations. For the one FLR event observed by FAST, the observed period is 13 min which is approximately 9 times longer than the 8 s predicted by the model. Another difficulty pertains to the role of anomalous resistivity, which is not calculated self-consistently in the model but appears to be essential in obtaining the high quality of agreement between theory and observations. Without anomalous resistivity, the field-aligned current density predicted by the model becomes too large by an order of magnitude and contracts to a much narrower spatial scale compared with observations [Lotko et al., 1998]. It is worth exploring whether there are alternative collisionless mechanisms that can support large parallel electric fields without requiring the current density profiles to be controlled and “flattopped” in time by anomalous resistivity.

In this Letter, we develop a collisionless four-field model for FLRs. Our model is essentially equivalent to the Hamiltonian four-field model derived by Hazelton et al. [1987] which we have generalized slightly to allow for spatial dependency in the equilibrium plasma density profile. The (dimensionless) four-field variables (defined below) are: the flux function \( A \) that determines the perturbed magnetic field, the perturbed electrostatic potential \( \phi \), the perturbed electron pressure \( p \), and the perturbed parallel ion speed \( v \). The four-field model includes collisionless terms in the generalized Ohm’s law, finite-Larmor-radius effects, and plasma compressibility including the divergence of parallel and perpendicular flows. The equations used by Streltsov and Lotko, which originated from Chmyrev et al. [1988] constitute a two-field model involving only the variables \( A \) and \( \phi \). These two-field equations can be deduced from the four-field equations if certain couplings that exist rigorously between the four-field variables are neglected.

The four-field model provides possible remedies for the discrepancies between the Streltsov-Lotko model and observations. As we demonstrate below, the shear-
Alfvén wave is coupled to the slow wave in the four-field model. This coupling, which is lost in the two-field model, implies that the eigenfrequencies of FLRs can vary over a wider range, permitting the realization of lower values than are closer to the observed frequencies. The linear eigenmode spectrum obtained from the four-field equations is richer than that in the two-field system, and shows significant differences in the spatial structure of the electric field, current density, and ion flow perturbations. We show it is possible to realize large parallel electric fields in the four-field model without necessarily generating the large parallel current densities that motivated Lotko et al. [1998] to invoke an anomalous parallel resistivity.

The four-field model is similar in approach but significantly different in content from a recent formulation of reduced equations for FLRs due to Frycz et al. [1998], which has been used by Rankin et al. [1999] to investigate large-amplitude density fluctuations in an auroral cavity. It can be shown easily that for the equilibrium model considered in this paper, there is no coupling between the shear-Alfvén and slow modes in the reduced equations of Frycz et al. [1998].

The Four-Field Model

The four-field model [Hazeltine et al., 1987] was originally derived from two-fluid equations for describing the dynamics of low beta (less than unity) tokamak plasmas. The total magnetic field \( \mathbf{B} \) is represented as \( \mathbf{B} = \mathbf{B}_0 + \mathbf{B}_E + \mathbf{E} \times \mathbf{A}_L(x, z, t) \), where \( \mathbf{B}_0 = \beta \mathbf{B}_0 \) is the equilibrium magnetic field, assumed to be spatially uniform, and the flux function \( \mathbf{A}_L(x, z, t) \) determines the fluctuating magnetic field \( \mathbf{B}_L \) perpendicular to \( \mathbf{B}_0 \). The system is bounded at \( z = -L \) and \( L \) by perfecting conducting walls, representing the northern and southern ionospheres. The perturbed ion velocity transverse to the equilibrium magnetic field is written \( \mathbf{V}_I = \dot{z} \times \nabla \phi_I \), where \( \phi_I \) is the stream function for the velocity, \( \nabla = \hat{x} \partial / \partial x \) is the gradient operator perpendicular to the equilibrium magnetic field, and \( \nabla \parallel = \partial / \partial 
abla \parallel \) is the projection of the gradient operator parallel to \( z \). Following Hazeltine et al. [1987], and other references cited therein, we can derive the following four-field equations:

\[
(1 - d^2 A^2 V^2) \frac{\partial A}{\partial t} + (1 - 4 \rho^2 V^2) \nabla \nabla \phi \\
- \frac{1}{2} d(1 + \rho) V A \nabla \nabla \phi = 0 ,
\]

\[
\frac{\partial}{\partial t} \left( \nabla \perp \cdot \left( \frac{1}{V^2} \nabla \phi \right) \right) + \left( 1 + 4 \rho^2 V^2 \right) \nabla \nabla \phi = 0 ,
\]

\[
\frac{\partial \phi}{\partial t} + \beta \nabla \nabla \phi = 0 ,
\]

\[
\frac{1}{V^2} \frac{\partial V}{\partial t} = \frac{1}{2} (1 + \rho) \nabla \nabla \phi - \frac{1}{4} \beta \tau \nabla \nabla \phi = 0 .
\]

In (2)-(5), the independent variables are dimensionless by redefining \( x \rightarrow Lx, y \rightarrow Ly, z \rightarrow Lz, \)

t \rightarrow t L / V_A, \)

where \( V_A = B_0 / (4 \pi n_0 m_0) \) is the equilibrium Alfvén speed defined with a constant reference density \( n_0 \) for electrons as well as ions (in a hydrogen plasma). The dimensionless perturbed physical variables are defined as follows: \( \mathbf{A} = A_1 / (B_0 L), \) \( \phi \equiv \phi_1 / (V_A L), \) \( p \equiv p_1 / (B_0^2 / 8 \pi), \) and \( v \equiv v_1 / V_A, \) where \( \phi_1 \) is the stream function, \( p_1 \) is the perturbed electron pressure, and \( v_1 \) is the ion flow parallel to the background uniform magnetic field. The dimensionless equilibrium parameters are the skin depth \( d = c / [e p_1 L] \), where \( s \) stands for species (e for electrons, i for ions), \( c \) is the speed of light, \( \omega_p = [4 \pi n_0 e^2 / m_i] \) is the Langmuir oscillation frequency for species \( s \), the electron beta \( \beta_e = 8 \pi n_0 T_e / B_0^2, \) and \( \tau = T_i / T_e \) is the ratio of the equilibrium ion and electron temperatures, assumed to be constant, and \( p_i \) is the ion Larmor radius, which can be written \( p_i = (\beta_e d_i \tau) / (1 + \beta_e d_i \tau) \). The perturbed electrostatic potential \( \phi_1 \) is cast in the dimensionless form \( \phi_1 \equiv \phi_1 / (V_A L) \) and related to the dimensionless stream function by the equation (1) \( \rho^2 \nabla _\perp \phi = \phi_1 + d \tau p / 2 \). Note that the constant Alfvén speed parameter \( V_A \) is distinct from the spatially dependent dimensionless Alfvén speed function \( V_A(x, z) \equiv [n_0 / n(x, z)]^{1/2} \), where \( n_0(x, z) \) is the background density profile for ions.

It is instructive to consider the conditions under which (2)-(5) reduce to the two-field equations used by Streltsov and Lotko [1996] (hereafter, referred to as SL96). If we arbitrarily set the parallel ion flow \( v \) to zero in (2)-(4), they decouple from (5) and become a closed set. However, (5) continues to impose an independent constraint on the perturbed pressure \( p \), one that is not necessarily compatible with \( p \) calculated from (2)-(4). In other words, (2)-(5) become overdetermined. This happens because the neglect of the parallel ion flow \( v \) is not internally consistent and violates parallel momentum balance of the plasma fluid.

Even though it is not correct to do so, let us examine the consequences of ignoring the parallel ion flow \( v \) and (5). In addition, we drop the finite-Larmor-radius correction terms, that is, the second term on the left-hand side of each of (2) and (3) and set the parameter \( \tau \) equal to zero. It is then possible to eliminate \( p \) between (2)-(4), and obtain precisely (10) and (11) in SL96:

\[
(1 - d^2 A^2 V^2) \frac{\partial A}{\partial t} + \nabla \nabla \phi = 0 ,
\]

\[
\frac{\partial}{\partial t} \left( \nabla \perp \cdot \left( \frac{1}{V^2} \nabla \phi \right) \right) + \nabla \nabla \phi = 0 .
\]

Equations (6) and (7), which specify the time-evolution of the two scalar fields \( \phi \) and \( A \), are referred to in this Letter as the two-field equations.

Dispersion Relations

We now compare the linear dispersion relations contained in the two-field equations (6) and (7) with the four-field equations (2)-(5). We assume, for the moment, that the background density is uniform (that is, \( V_A(x, z) \equiv 1 \)), and write the linear perturbations in the Fourier form \( \exp \{ i (k_\perp x + k_\parallel z - \omega t) \} \). The two-field equa-
tions (6) and (7) then yield the (dimensional) dispersion equation
\[ \omega^2 = k_{||}^2 V_{A0} \frac{1 + (\beta_s k_{||}^2 d_i^2/2)}{1 + k_{||}^2 d_e^2}. \] (8)

The dispersion equation (8) corresponds essentially to the shear-Alfvén branch (1), modified by kinetic and inertial effects that make it dispersive.

The Fourier analysis of (2)-(5) yields, in matrix form, the linear algebraic equations
\[
\begin{pmatrix}
\frac{\omega^2}{k_{||}^2} & 0 & \frac{-\beta_s^2}{4d_i} \\
(1 + k_{||}^2 d_i^2) & \frac{4d_i(1 + \tau)}{2} & 0 \\
0 & k_{||}^2 \beta_s d_i & \frac{-\beta_s}{k_{||}^2} \\
-\frac{k_{||}^2 \beta_s d_i}{4} & 0 & -(1 + \tau) \frac{\omega^2}{k_{||}^2}
\end{pmatrix}
\begin{pmatrix}
\phi \\
A \\
P \\
v
\end{pmatrix}
= 0
\]

(9)

which has nontrivial solutions if and only if the (4 \times 4) determinant is zero. Neglecting \(O(k_{||}^4 \beta_s^4)\) terms and relation \(\beta_s^2 \ll 1, \tau \leq O(1)\), we obtain the two roots
\[
\frac{(\omega^2/k_{||}^2 V_{A0})^2}{2(1 + k_{||}^2 d_i^2)} \left[ 1 + \frac{1}{2} \beta_s(1 + \tau) \right] + \left( \frac{1 + \tau}{2} + \frac{\beta_s \tau}{16} \right) k_{||}^2 \beta_s^2 \\
\times \left[ 1 - \beta_s(1 + \tau) + \frac{1 + \tau}{\tau} + \frac{\beta_s \tau}{8} \right] + \frac{\beta_s(1 - \tau^2)}{2\tau} k_{||}^2 \beta_s^2 \right]^{1/2}.
\] (10)

In the dispersion relation (10), the shear-Alfvén mode is coupled to the slow mode, allowing for a richer frequency spectrum and the excitation of lower frequencies. We demonstrate this below by numerical eigenmode calculations.

In contrast with the dispersion relation (10), the dispersion relation (8) follows from the linear equations (9) if we set the parallel ion flow \(v\) and the parameter \(\tau\) to zero, and neglect, furthermore, the finite-Larmor-radius corrections in the remaining (3 \times 3) determinant. The neglect of the parallel ion flow in the two-field model leads to the loss of coupling to the slow wave, and thus limits the range of realizable eigenfrequencies for FLRs.

Numerical One-Dimensional Eigenmodes

To obtain the eigenfrequency and parallel wave structure of a stationary dispersive FLR, we follow SL96 and neglect, as a first approximation, the spatial inhomogeneity of the plasma transverse to equilibrium magnetic field \(B_0\). We write \(\partial / \partial z = ik_{||}\) and seek normal mode solutions \((\phi, p) \propto e^{i\omega t}\) and \((A, v) \propto e^{-i\omega t}\).

The resultant one-dimensional equations subject to the boundary conditions for a perfectly conducting ionosphere \((\phi - \nu = 0)\), determines the eigenmodes. As in SL96, because the equilibrium magnetic field is a constant, we use a z-dependent density profile to determine the spatial dependence of the Alfvén speed function \(V_A(z)\). The parametrization of this density profile is discussed in detail in SL96. For the numerical results presented below we choose the SL96 profile with \(\varepsilon_1 = 1 R_E, \varepsilon_2 = 4 R_E\), and keep all other parameters the same. We recognize that this density profile is not realistic, but because one of the main objectives of this Letter is a comparison of the predictions of the two-field and four-field models, we use the same density profiles as SL96 in our two-field and four-field computer runs. Other relevant physical parameters in (11)-(14) are \(k_{||} d_i \approx 0.3, \tau \approx 10, \beta = 0.1, d_e \approx 1.1 \times 10^{-3}, \) and \(d_i \approx 4.6 \times 10^{-2}\).

Equations (11)-(14) are linear, and there is an arbitrariness in the overall normalization of the perturbed quantities. For comparison with SL96, we choose the maximum magnitude of the (dimensional) parallel electric field, whereupon the normalization of the other perturbed quantities are fixed by the equations.

In Figure 1 we plot the fundamental eigenmode profiles as a function of \(z\) from the four-field (solid line) and two-field (dotted line) models for the perturbed perpendicular electric field \(E_z\) (Figure 1a), field-aligned current density \(J_z\) (Figure 1b), and parallel electric field \(E_z\) (Figure 1c). In each of the three plots, the scale on the left (right) corresponds to the four-field (two-field) model. Although the parallel electric field is comparable in magnitude in the four-field and two-field models, the profiles are significantly different. The perpendicular electric fields are larger in the two-field model by nearly an order of magnitude. The spatial distribution of the field-aligned current density in both models is quite similar, but the magnitude of the current density in the four-field model is smaller than that in the two-field model by well over an order of magnitude.

![Figure 1. Fundamental eigenmode profiles of (a) \(E_z\), (b) \(J_z\), and (c) \(E_z\) as a function of \(z\), the distance along an equilibrium field line, from the four-field (solid line) and SL96 (dotted line) models. The left (right) vertical scale corresponds to the four-field (two-field) models. The relevant physical parameters are given in the text.](image-url)
The eigenfrequency in the two-field model (for the given density profile and physical parameters) is \( \omega_{\text{SL}} \approx 0.26 k_1 V_{A10} \). In contrast, the eigenfrequency in the four-field model is \( \omega \approx 0.04 k_1 V_{A10} \), smaller than the two-field result by nearly a factor of seven. The eigenfrequency in the two-field model does not change much with \( k_1 \), but that in the four-field model is much more sensitive and can vary over a wide range, producing a richer spectrum.

As mentioned above, a crucial difference between the two models is the role played by the parallel ion flow \( v_p \) neglected in the two-field model. In Figure 2, the solid line is the profile of \( v \) for the four field model, normalized by the equilibrium Alfvén speed function. The dotted line represents what \( v \) would have been in the two-fluid model (presented in SL96) if it were neglected in (2)-(4), but calculated from the decoupled equation (5). This serves as a consistency check as to whether or not the parallel ion flow can be ignored. That \( v \) is not small even in the two-field approximation is obvious by inspection of Figure 2, and reinforces our claim that the neglect of \( v \) is not justified. The dotted line, which describes the parallel ion flow in the SL96 model, has an apparent unphysical discontinuity, caused by the lack of internal self-consistency in the Streitsov-Lotko model. The velocities in Figure 2 for both models are larger than is typically observed, and this is because the SL96 density profiles are somewhat artificial, especially near the ionospheric boundaries.

In conclusion, we suggest that the four-field model is a useful point of departure for the study of field-line resonances in collisionless magnetospheric plasmas. The present model builds on the accomplishments of the Streitsov-Lotko model and brings theory qualitatively closer to observations in two significant ways: (i) it allows for the excitation of a richer spectrum of FLR eigenfrequencies due to coupling to the slow mode, including frequencies that are lower than predicted by the shear-Alfvén dispersion relation, and (ii) it produces large parallel electric fields without field-aligned current densities that exceed by far those that are observed.

The present calculation is a first step, and needs to be extended to include the Earth's dipole field and more realistic density profiles before quantitative comparisons with observations can be made. Our main objective in this paper has been to point out the importance of the coupling between the shear-Alfvén and slow wave due to the mediation of parallel ion flow in the four-field model. It thus represents a fundamentally important physical effect that must be retained in a more elaborate model of FLRs which employs realistic magnetic field and pressure profiles. The results from such a model will be published in a future paper.

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