Analytic calculation of the nonzero fast wave reflection coefficient from the tunneling equation with absorption

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An analytic series expression of the nonzero fast wave reflection coefficient has been found from the tunneling equation which models the wave propagation in weakly inhomogeneous media near a back-to-back resonance-cutoff pair. Explicit calculations are done for both second ion cyclotron and second electron cyclotron harmonic cases. Numerical values of this expression are compared to those calculated by solving an integral equation iteratively. They agree well with each other whenever both the series and the iteration converge. This result is based on a new method of evaluating integrals derived from the integral equation without having to solve the equation itself. This new method may have more applications and deserves further research and development.

I. INTRODUCTION

Waves propagating in weakly inhomogeneous media through a back-to-back resonance-cutoff region, in general, experience effects of transmission, reflection, and mode conversion with or without absorption. These phenomena can be modeled by different kinds of high order ordinary differential equations. One of the simplest kinds of these is a fourth order equation of the form (e.g., Refs. 1–4)

\[ f'' + \lambda z (f'' + f) + \gamma f = 0, \tag{1} \]

with \( \lambda \) and \( \gamma \) being real constants. We call this the tunneling equation without absorption. This equation can model many different physical situations. We will explicitly consider two of them: (i) second ion cyclotron harmonic with \( \gamma > -1 \); (ii) second electron cyclotron harmonic with \( \gamma < -1 \). We can generalize (1) to include localized absorption.\(^3\)

\[ \psi'' + \lambda^2 z (\psi'' + \psi) + \gamma \psi = h(z) (\psi'' + \psi), \tag{2} \]

where \( h(z) \) is the absorption function which generally must fall off at least as fast as \( z^{-1} \) as \( |z| \to \infty \). Analytic expressions for the scattering parameters, i.e., transmission, reflection, and mode conversion coefficients, for Eq. (1) have long been known. Solutions for Eq. (2), and thus scattering parameters, can be found by solving an integral equation iteratively, making use of the numerical solutions of Eq. (1). These scattering parameters are expressed in terms of integrals involving \( h(z) \), solutions of Eqs. (1) and (2) along the \( z \)-axis.

It has been pointed out that odd order derivatives should be added to Eqs. (1) and (2), with a much more complicated expression in the right hand side of Eq. (2), in order to conserve energy.\(^3\) However, from previous numerical results, the scattering parameters found by both sets of equations are close, especially for the reflection coefficient \( R_3 \).\(^3\) So we will use the simpler equations for our analysis. We believe that the method we develop here should also work for the energy conserving equations, since the two sets of equations are very similar.

Recently, by extending solutions of Eqs. (1) and (2) into the complex \( z \)-plane, some of these integrals were shown to be zero identically.\(^6\) Thus, fast wave transmission coefficients (which are equal to each other) from both sides and the fast wave reflection coefficient (which is identically zero) from the side which encounters the resonance before the cutoff, were shown to be independent of absorption. However, analytic expressions for other scattering parameters which do change with absorption have been unknown so far. Here, we go further along this direction and develop a new method to calculate some of those nonzero integrals analytically.

We note that the above method is not the only approach used to solve the mode conversion problem. Other methods have been developed, including, e.g., direct numerical integration,\(^7\) finite element,\(^8\) finite difference,\(^9\) phase space method,\(^10\) and order reduction.\(^11-14\) Partial\(^13\) or approximate expressions\(^10,14\) of the scattering parameters have also been reported. However, we will concentrate on the above integral equation method of solving Eq. (2) here. We emphasize that the analytical expression for the reflection coefficient given here is exact in the sense that no further approximation is used after we accept Eqs. (1) and (2), which are in a general mathematical form that may represent many other physical situations. In all other semi-analytic methods, additional approximations are made before evaluating the reflection coefficient.

We have been successful in calculating the other fast wave reflection coefficient \( R_2 \), which is nonzero, for the above two cases (i) and (ii). It seems that this method, with further development, may also be useful in calculating other scattering parameters or integrals of similar nature for other problems.

In the next section, we will review the integral equation method and thus the integral expressions of the scattering parameters. In Sec. III, we will discuss the basic ideas of this method and show explicitly how to calculate \( R_3 \) analytically for the two cases. We will compare results from this method to those calculated by the integral equation method in Sec. IV.
II. INTEGRAL EQUATION

Equation (1) can be solved exactly by using the method of Laplace:

\[ f_j(z) = c_j \int_{\Gamma_j} \left[ \exp \left( z g(k) \right) \right] dk, \quad j = 1, 2, 3, 4, \quad (3) \]

where the \( \Gamma_j \) are contours in the complex \( k \)-plane which end at infinity with approach angles of \( \pi/6, 5\pi/6, \) or \( 3\pi/2, \)

\[
\begin{align*}
0 & \quad T_1 & 0 & \quad C_{14} & \quad f_+ & \quad -i & \quad z & \quad f_1 & \quad 1 & 0 & \quad C_{13} & 0 & \quad f_+ \\
1 & \quad R_2 & 0 & \quad C_{24} & \quad f_- & \quad -i & \quad z & \quad f_2 & \quad 0 & 1 & \quad C_{23} & 0 & \quad f_- \\
0 & \quad C_{32} & 0 & \quad C_{34} & \quad \sigma_+ & \quad f_3 & \quad 0 & \quad C_{31} & 0 & \quad R_3 & 1 & \quad f_3 \\
0 & \quad C_{42} & 1 & \quad R_4 & \quad \sigma_- & \quad f_4 & \quad 0 & \quad C_{41} & 0 & \quad C_{43} & 0 & \quad f_- \\
\end{align*}
\]

for \( \gamma > -1, \) and

\[
\begin{align*}
0 & \quad T_1 & 0 & \quad C_{14} & \quad f_- & \quad -i & \quad z & \quad f_1 & \quad 0 & 1 & \quad C_{13} & 0 & \quad f_- \\
R_2 & 0 & \quad C_{24} & \quad f_+ & \quad -i & \quad z & \quad f_2 & \quad 1 & 0 & \quad R_4 & 0 & \quad C_{23} & 0 & \quad f_+ \\
C_{31} & 0 & \quad C_{34} & \quad \sigma_+ & \quad f_3 & \quad 0 & \quad C_{32} & 0 & \quad R_3 & 1 & \quad f_3 \\
C_{41} & 0 & \quad 1 & \quad R_4 & \quad \sigma_- & \quad f_4 & \quad 0 & \quad C_{42} & 0 & \quad C_{43} & 0 & \quad f_- \\
\end{align*}
\]

for \( \gamma < -1, \) with scattering parameter given by

\[
S_{ij} = \begin{pmatrix}
R_1 & T_1 & C_{13} & C_{14} \\
T_2 & R_2 & C_{23} & C_{24} \\
C_{31} & C_{32} & R_3 & C_{34} \\
C_{41} & C_{42} & C_{43} & R_4
\end{pmatrix} S^{(0)}_{ij} = \begin{pmatrix}
0 & g & -\xi & 0 \\
g & \xi & g^2 & \xi \\
-\xi & g^2 & g & \xi \\
0 & \xi & g & 1/2
\end{pmatrix}
\]

where \( g = \exp(-\eta), \quad \eta = \pi |\alpha|, \quad \xi^2 = 1 - g^2, \) and

\[
f_\pm = -\frac{\pi e^{-\eta/2}}{2\pi i \sqrt{z}} \exp \left[ \pm i \frac{2}{3} \lambda z^{1/2} + \frac{1}{2} \ln |z| \right] \int_{\Gamma_j} \left[ \exp \left( z g(k) \right) \right] dk,
\]

\[
s_\pm = \text{sgn}(\alpha) \frac{\sqrt{\pi}}{\sqrt{\lambda \sqrt{2\pi} z^{3/4}}} \exp \left[ \pm i \frac{4}{3} \lambda z^{1/2} - \frac{1}{2} \ln |z| \right],
\]

\[
\sigma_+ = -\frac{i \sqrt{\pi}}{\sqrt{\lambda \sqrt{2\pi} z^{3/4}}} \exp \left[ \frac{2}{3} \lambda |z|^{1/2} - |z|^{1/2} \right],
\]

\[
\sigma_- = -\text{sgn}(\alpha) \frac{\sqrt{\pi \lambda}}{\sqrt{\lambda \sqrt{2\pi} z^{3/4}}} \exp \left[ \frac{2}{3} \lambda |z|^{3/2} - |z|^{1/2} \right].
\]

Only \( f_1, f_2, f_3 \) are physically allowed in an unbounded region. Using these we can find an integral equation that solves Eq. (2): \(^5\)

\[
\Psi_k = f_k = \frac{1}{8} \left[ f_2 I_{1k} + f_3 I_{0k} + f_1 I_{2k} + f_0 I_{4k} \right], \quad \gamma > -1,
\]

\[
\Psi_k = f_k = \frac{1}{4} \left[ f_1 I_{2k} + f_3 I_{0k} + f_2 I_{1k} + f_0 I_{3k} \right], \quad \gamma < -1,
\]

where

\[
g(k) = -i k + \frac{i}{2} \left[ \frac{k^3}{3\lambda^2} + \frac{k}{\lambda^2} + (i-\alpha) \ln(k+1) \right] + (i+\alpha) \ln(k-1),
\]

where \( \alpha = (1+\gamma)/2\lambda^2. \) With a suitable choice of constants \( c_j, \) we can express the asymptotic behavior of \( f_j \) (note that the definitions in this paper are in general different from those in previous literatures\(^4\)) as

\[
I_{jk} = \pm \frac{1}{2\pi i \lambda^2} \int_{z=\infty}^{z=-\infty} F_j(y) h(y) \Psi_k(y) dy,
\]

with

\[
F_j = f_j' + f_j, \quad \Psi_j = \psi_j' + \psi_j,
\]

and \( f_0 = f_3 - \xi f_1, \) for \( \gamma > -1, \) \( f_0 = g(\xi f_2 + g f_3), \)

\[
f_5 = f_4 - \xi f_1 / g, \quad \text{for} \quad \gamma < -1. \]

Equation (7) can be solved iteratively using \( \psi_k = f_k \) as the first trial functions with \( f_k \) calculated numerically by Eq. (3). This can be done for the three physical solutions, \( k = 1, 2, 3, \) if \( h(z) \) fall off at least as fast as \( z^{-1} \) as \( z \to \infty. \) For the \( k = 4 \) solution, this can be done only if \( h(z) \sigma_+(z) \to 0 \) fast enough as \( z \to -\infty. \) Numerically, this method converges in general, but may become divergent when the absorption is very large. After solving Eq. (7), scattering parameters can be found, making use of Eqs. (4)-(6), by

\[
S_{jk} = S_{jk}^{(0)} = I_{jk} \quad \text{for} \quad \pm (\gamma+1) > 0,
\]
\[ I_{jk} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} F_j(z) h(z) \Psi_k(z) \, dz, \]

where we have already used the fact that \( I_{ik} = I_{kj} \). In particular, the integral expression for \( R_2 \) is

\[ R_2 = \frac{1}{\gamma} \int \mathcal{F}(z) \mathcal{G}(z) \, dz, \]

In the next section, we will show how to calculate this integral analytically, without the need of solving Eq. (3) or Eq. (7) numerically.

### III. INTEGRATION

To perform the integration of \( I_{22} \), we must first specify the absorption function \( h(z) \). For case (i), the second ion cyclotron harmonic with \( \gamma > -1 \),

\[ h(z) = \lambda_2^2 k \left[ \xi - 1/Z(-\xi) \right]; \]

and for case (ii), the second electron cyclotron harmonic with \( \gamma < -1 \),

\[ h(z) = \lambda_2^2 k \left[ \xi - 1/F_{7/2}(\xi - 7/2) \right], \]

with the plasma dispersion function \( Z(\xi) = i \frac{\sqrt{\pi}}{\xi} F(\xi) \), where \( \omega \) is the error function for complex argument, and \( F_{7/2} \) being the relativistic plasma dispersion function, and \( \xi = (z - z_0)/k \), \( z_0 = -\gamma/\lambda^2 \), where \( k \) is a real parameter characterizing the strength of absorption. Note that both \( Z(\xi) \) and \( F_{7/2}(\xi) \) are analytic function and have zeros only in the lower half \( \xi \)-plane. Also, solutions \( F_k \) and \( \Psi_k \) have been shown to be analytic everywhere. Then, following Ref. 6, we can change the path of integration of \( I_{jk} \), defined in Eq. (9), to the semicircles \( C_\pm \): \( z = R \exp(i\theta) \) with \( R \to \infty \) and \( \theta \) from \( -\pi \) to 0 for case (i), see Fig. 1, and \( \theta \) from 0 to \( \pi \) for case (ii), see Fig. 2. In Ref. 6, solutions \( f_k \) and \( \psi_k \) have been extended to the complex \( z \)-plane, and it was shown that on these new contours \( C_\pm \), both \( f_1 \) and \( \psi_1 \), \( \exp(\pm i\zeta) \) and are thus exponentially small for both cases of \( \gamma > 0 \), and that \( f_2 \) and \( \psi_2 \), \( \exp(\pm i\zeta) \) are thus exponentially large. It was then concluded that \( I_{11} = I_{12} = I_{21} = 0 \), which means that \( T_1 = T_2 = 0 \), and \( R_1 = 0 \), independent of absorption. Our approach here to calculate \( I_{22} \) is to expand the integrand \( F_2 h(\Psi_2) \) in an asymptotic series on \( C_\pm \).

Since \( |z| \to \infty \) on the contour, all terms in this asymptotic series may be kept. Although the integrand is in general a divergent series in \( z^{-n} \) for any finite \( z \), the series obtained after integration may become convergent for certain absorption function \( h(z) \). First of all, it is easy to expand \( h(z) \) in asymptotic series,

\[ h(z) = \sum_{n=1}^{\infty} \frac{h_n}{\gamma^n}, \quad \text{for } \pm (\gamma + 1) > 0, \]

where \( \gamma = z - z_0 \). This can be done by inverting the two asymptotic series,

\[ Z(\xi) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{\Gamma(1/2)\xi^{2n}}, \]

and

\[ F_2(\xi) = \sum_{n=0}^{\infty} \frac{(\gamma+1)^n \Gamma(n+q)}{\Gamma(q)\xi^{q_2}}, \]

on the path of integration \( C_\pm \). First, we need to shift the argument of the \( F_{7/2} \) function in Eq. (10) from \( \xi - 7/2 \) to \( \zeta \). Let,

\[ F_2(\xi - q) = \sum_{n=1}^{\infty} \frac{A_n(q)}{\xi^n}; \]

The coefficients \( A_n(q) \) can be found, using Eq. (13), by

\[ A_n(q) = \frac{1}{\Gamma(q)} \sum_{m=1}^{\infty} (-1)^m \Gamma(q+m-1) C_{n-1}^{m} q^{-m}, \]

where \( C_{n}^{m} = n!/m!(n-m)! \) is the binomial coefficients. The first few values of \( A_n \) are \( A_1 = 1 \), \( A_2 = 0 \), \( A_3 = q \). Then we can invert \( F_2 \) by,

\[ \frac{1}{F_2(\zeta-q)} = \zeta \left[ 1 + \sum_{n=2}^{\infty} \frac{C_n(q)}{\zeta^n} \right], \]

where

\[ C_{2m+1/2,1/2}(q) = \sum_{n=1}^{m} \left[ -A_3(q) \right] D_{2(m-n)+1/2,1/2}(q,n), \]
for \((y + 1) > 0\), where

\[
\Psi_{\pm}(y) \equiv c_{\pm} \sum_{n=1}^{\infty} \frac{a_n^\pm}{y^n} e^{\pm i(n-\alpha y)},
\]

\[
F_{\pm}(y) \equiv c_{\pm} \sum_{n=1}^{\infty} \frac{b_n^\pm}{y^n} e^{\pm i(n+\alpha y)},
\]

with

\[
c_{\pm} = \frac{-\eta^\pm m/2}{\xi \alpha \Gamma(\pm i \alpha)} \exp \left[ \pm i \left( 4 + \alpha \ln 2 + 2 \right) \right].
\]

The coefficients \(a_n^\pm\) and \(b_n^\pm\) can be found by substituting the asymptotic series (17), making use of Eq. (8), into the two differential equations (1) and (2), requiring that all terms vanish and that they satisfy boundary conditions Eq. (4) or (5). The result is

\[
a_{n+1}^\pm = a_{n+1}^\pm - a_{n+3}^\pm, \quad n > 1,
\]

\[
a_{n,1}^\pm = \pm \frac{1}{2i \alpha n} \left[ (\alpha \pm i(n-1)) \left( a_{n-1,4}^\pm + a_{n-1,3}^\pm + (1 + \gamma) a_{n-1,2}^\pm + \lambda^2(\alpha \pm i n) a_{n-1,1}^\pm \right) - \sum_{m=1}^{n-1} h_{n-m}^m a_m^\pm \right],
\]

\[
a_{n,k}^\pm = a_{n,k-1}^\pm + [\alpha \pm i(n-1)] a_{n-1,k}^\pm, \quad k = 2, 3, 4,
\]

and \(a_{0,k}^\pm = 1\). The calculation for \(a_n^\pm\) is similar to Eqs. (18), with \(h_n^\pm = 0\). Substituting Eqs. (11) and (17) into Eq. (16), we get

\[
2\pi i \lambda^2 I_{22} = c_{\pm} \sum_{n=1}^{\infty} \gamma_n^\pm \int_{C_\pm} \frac{1}{y^n} e^{\pm 2i(n-\alpha y)} dy,
\]

for \((\gamma + 1) > 0\), where

\[
\gamma_n^\pm = \sum_{m=1}^{n-2} \beta_n^m h_n^m, \quad n > 3,
\]

\[
\beta_n^\pm = \sum_{m=1}^{n-1} a_{n-m}^\pm a_n^\pm, \quad n > 2.
\]

To evaluate the \(y\)-integrals in Eq. (19), we change the integration contour again. Now, there is only one pole \(y = 0\) in the integrand, and there is also a branch cut. We choose the branch cut to be from \(y = 0\) to \(\pm i \infty\) for the two cases (see Figs. 1 and 2). Then we change the integration contours to go along the semicircles with infinite radius at the other side of the real axis, opposite to \(C_{k}\), and go around the branch cut (see Figs. 1 and 2) so that the analytic continuation is still valid. The contributions from integrating along the two quarters of the semicircles are zero due to the \(\exp \pm 2i y\) factor. Integration along the path going around the branch cut can be found by making use of the Hankel's contour integral,15

\[
\int_{C_H} (e^{-i \pi t})^{-z} e^{-it} dt = -2\pi i \frac{\Gamma(z)}{\Gamma(z)},
\]

where the contour \(C_H\) starts at \(t = \infty \exp(0i)\), come along the positive real axis, turn around the origin counterclockwise, then go back along the positive real axis to \(t = \infty \exp(2\pi i)\). The \(y\)-integral in Eq. (19) can be changed to this form by means of a variable transformation \(t = 2y \exp(3\pi i/2)\) for \(\gamma > -1\), and \(t = 2y \exp(\pi i/2)\) for \(\gamma < -1\). Thus, we can express \(I_{22}\), or \(R_2\) in a series expression,

\[
R_2 = \gamma_n^\pm \int_{C_\pm} \frac{1}{y^n} e^{\pm 2i(n-\alpha y)} dy = \sum_{n=1}^{\infty} \gamma_n^\pm \frac{\pm 2i}{\xi \alpha \Gamma(\pm i \alpha)} \frac{\Gamma(n+2 \alpha)}{\Gamma(n+2 \alpha)},
\]

for \((\gamma + 1) > 0\). This expression is analytic in the sense that all coefficients \(\gamma_n^\pm\) can be calculated exactly by some algebraic recurrence formula, although they are in general rather lengthy to be written out explicitly. This series is useful only if it is convergent, unless we know how to sum it analytically. However, due to the complicated nature of the coefficients, it is beyond the scope of this paper to study the convergence analytically. In the next section, we will present numerical results showing that it does converge, in many cases, to numerically the same values found by solving the integral equation (7) iteratively.

IV. COMPARISONS

We do our calculations with a deuterium plasma. Table I compares calculations of \(R_2\) for case (i), the second ion cyclotron harmonic with \(\gamma > -1\), for plasma parameters: electron density \(n_e\), characteristic length of the inhomogeneity \(L\), magnetic field strength \(B\), and plasma tem-
Table I. Case (i) with \( n_e=2 \times 10^{20} \, m^{-3} \), \( L=3 \, m \), \( B=3 \, T \), \( T=1000 \, eV \).

<table>
<thead>
<tr>
<th>( k_\parallel (m^{-1}) )</th>
<th>( \kappa )</th>
<th>( P_a(%) )</th>
<th>( P_o(%) )</th>
<th>( q_a )</th>
<th>( q_o )</th>
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<tbody>
<tr>
<td>1</td>
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<td>60.498</td>
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<td>0.9893</td>
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</tr>
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</tr>
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</tr>
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<td>(2.1e-7)</td>
<td>(2.1e-7)</td>
<td>0.794</td>
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</tbody>
</table>

Temperature \( T \), for different parallel wave vectors \( k_\parallel \). \( P_a \) is \(|R_2|^2 \) calculated by solving the integral equation (7) iteratively. \( P_o \) is that calculated by the series in Eq. (20). Table II compares these for case (ii), the second electron cyclotron harmonic with \( \gamma=-1 \), where \( X \) is the square of the ratio of the plasma frequency to the wave frequency.

Previously, it was found numerically that

\[ R_2 \approx e^6 \exp(-q\kappa^2), \]

\[ q \approx 1 \quad \text{for case (i)}, \]

\[ q \approx 7 \quad \text{for case (ii)}. \] (21)

We also list the \( q_a \) and \( q_o \) values, calculated by each method, defined by \( P_a=g^6 \exp(-2q_o\kappa^2) \) and \( P_o=g^6 \exp(-2q_a\kappa^2) \). They are also plotted in Figs. 3 and 4.

We first note that these values calculated by both methods agree with each other very well as long as both methods converge. We have also compared results from these two methods for many other cases over a broad range of parameters and they all agree very well. It has been known previously that the iteration method of solving Eq. (7) becomes divergent for large absorption (i.e., large \( \kappa \)) for both cases. It seems that the series method using Eq. (20) for case (i) is still convergent after the iteration method diverges. However, it also has numerical problems for large \( \kappa \), due to the fact that we can only sum a finite number of terms and that the computer has a limited power to handle large numbers. On the other hand, the series method becomes divergent faster than the iteration method for case (ii). Actually, it seems that there is a convergence radius of \( \kappa < 0.5 \) for the series. The reason for this difference is that the asymptotic series of the relativistic plasma dispersion function \( F_{\gamma/2} \) of Eq. (13) diverges faster than that of the plasma dispersion function \( Z \) of Eq. (12).

Second, we see that the approximation Eq. (21) is good, but not exact. Before the development of this series method, the numerical values found by solving the integral equation iteratively subject to many different kinds of numerical errors, which are difficult to analyze, within the complicated algorithm to calculate \( f_k \) and \( \psi_k \). Therefore, there was an uncertainty about whether Eq. (21) is really exact, and whether the deviations from the rule in the numerical results are just numerical errors. However, now that the series method is analytic in principle, its results are much more reliable and to the extent that they agree with those calculated by the iteration method, this uncertainty no longer exists. Moreover, the series method provides some analytic explanation of Eq. (21), at least in extreme

![FIG. 3. Plot of \( 1-q_a, 1-q_o \), and \( 1-q_f \) versus \( \kappa \), where \( q_a \) and \( q_o \) are the values in Table I, and \( q_f \) are calculated by Eq. (22).](image)

![FIG. 4. Plot of \( 7-q_a, 7-q_o \), and \( 7-q_f \) versus \( \kappa \), where \( q_a \) and \( q_o \) are the values in Table II, and \( q_f \) are calculated by Eq. (23).](image)
cases. Let us consider large $\lambda^2$, small $\eta$, and small $\kappa$ cases. It is not hard to see that in these cases, we only need to keep the $n = 3$ terms in the series in Eq. (20). Note that $\gamma_1^2 = 2\alpha^2\lambda^2\kappa^2$, and $\gamma_2^2 = 14\alpha^2\lambda^2\kappa^2$, and using approximations

$$\Gamma(\pm ia) \approx \pm 1/ia, \quad \Gamma(3 \pm 2ia) \approx 2, \quad g \approx 1, \quad \xi \approx 2\eta,$$

we have

$$R_2 \approx \xi^2(1 - \kappa^2) \approx \xi^2 \exp(-\kappa^2) \quad \text{for case (i)},$$

$$R_2 \approx \xi^2(1 - 7\kappa^2) \approx \xi^2 \exp(-7\kappa^2) \quad \text{for case (ii)},$$

which is just Eq. (21). From numerical calculations, this approximation is very good whenever the condition of large $\lambda^2$, small $\eta$, and small $\kappa$ is satisfied, although it is also good for some more general cases.

Since Eqs. (21) are good approximations already, it is possible to find even better empirical formulas by fitting the results found by the series method. For the ion cases, we found that the $q$ factor can be approximated by the following expression, so that $|R_2| \approx \xi^2 \exp(-q_\kappa^2)$,

$$q_f = 1 - \eta \xi^2(a_0 + a_1\kappa^2),$$

(22)

where

$$a_0 = 0.0065(1 + 0.131w + 0.103w^2),$$

$$a_1 = \exp(17.5e^{-8.5w} - 7 + 0.3w),$$

$$v = 3.085(1 - 0.1056w),$$

$$w = 100\beta_L,$$

and $\beta_L = \lambda^2 \kappa_T^2/B_{\rho}^2$ is the usual $\beta$ factor of the plasma. This approximation has been compared with the results of the series method, for the range $\kappa^2 < 4$, so that $|R_2|^2$ is at least greater than 0.04% of that without absorption, i.e., $\xi^2$, and for $k_2 < k_{\text{max}}/2$, where $k_{\text{max}}$ is the $k_z$ value when the X-mode is cut off. The error in $|R_2|$ rarely exceeds 1%, and this happens only for the cases with $w > 2$. In all cases, the error is smaller than 1% when $|R_2| > 1%$. Without this correction, i.e., simply using Eq. (21), the error in $|R_2|$ may exceed 15% for some cases. The $q_f$ factor calculated by Eq. (22) for the cases of Table I is also plotted in Fig. 3. The error between $q_f$ and $q_n$ seems large in Fig. 3 for some cases. However, we should note that actual difference is only about 2% in the $q_f$ factor for the worst cases, and that this happens only for $\kappa^2 > 4$.

For the electron cases, the approximation is

$$q_f = 7 - A[1 - e^{-2\kappa^2}],$$

so that $|R_2| \approx \xi^2 \exp(-q_\kappa^2)$, where

$$A = 3.602 + 2.413X,$$

$$a = -13.6 + 17.04/(1 - 2X)^{0.992}.$$ 

The comparison of this approximation with exact values is limited by the convergence of the series method. The range of $|R_3|$ is from 0.15% to 50% of $\xi^2$, the reflection coefficient without absorption. The range of the $X$ parameter is from 0.114 to 0.457. The range of the temperature $T$ is up to 850 eV. Within these ranges, the error of this approximation in $|R_2|$ is less than 0.5%. The $q_f$ factor calculated by Eq. (23) for the cases of Table II is also plotted in Fig. 4. We see that the approximation is good for the whole range of data shown on Fig. 4. The large difference between $q_\kappa$ and $q_f$ for the last data point is due to the error in $q_\kappa$, because the convergence of the series method is not good for $\kappa$ near 0.5. However, the agreement between $q_f$ and $q_n$ is still good, so the approximation Eq. (23) may still be good beyond the $\kappa < 0.5$ limit.

V. DISCUSSION

The fact that these two very distinct methods give essentially the same answers further confirms their own validity respectively, because if either one of these methods is wrong, the chances for their answers to agree with each other accidentally is extremely small.

The success of the series method is amazing and shows how powerful complex analysis can be. In the original form of the integral $I_{22}$ in Eq. (9), we needed to know the values of the solutions $F_2$, $\Psi_2$, and the absorption function $h$ along the real $z$-axis. The straightforward way to do this is first to find $F_2(z)$ by a numerical path integration in the complex plane using the method of Laplace like Eq. (3) with a complicated integration and error control scheme. Second, solve the integral equation (7) by doing numerical integrations iteratively along the $z$-axis, with proper endpoint corrections, to find $\Psi_2(z)$, making use of numerical methods to evaluate $h(z)$ for $z$ within different regions on the $z$-axis. Then find $I_{22}$ by numerical integration according to the definition Eq. (9). Now, the series method can do the same integration correctly without having to know $F_2$, $\Psi_2$, and even $h$ on the real $z$-axis. All it needs is its analytic properties in the complex $z$-plane and their asymptotic expansions for large complex $z$ values. Also, due to its less complicated algorithm, the series method usually runs much faster on a computer, if we do not care to know the solutions themselves. This powerful idea may have other applications to solve integrals of similar nature, which arise either from the tunneling problem or other physical problems.

The main difficulty the series method faces is that it does not converge for all cases. However, since it is essentially an analytical method, there is always a chance that we may find a way to sum the series analytically or to transform it into other convergent forms. Just like we can sum $1 + x + x^2 + x^3 + \cdots$ as $1/(1-x)$, although the series may be divergent itself. Note also that the iteration method also has divergence problems, but the chance to “sum” it analytically is much smaller, since it is basically numerical in nature.

We have also tried to apply this method to calculate the conversion coefficient $C_{13}$, but the resulting series is divergent. However, it is still possible that there is a way to overcome this difficulty.

We conclude that the series method has been successful in the calculations of the fast wave reflection coefficient from the tunneling equation and it has advantages in some
aspects. This also shows that the analytic method is powerful, and may have more applications and improvements.

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4D. G. Swanson, Plasma Waves (Academic, Boston, 1989).
5D. G. Swanson, Phys. Fluids 21, 926 (1978).