Simulations and Transport Models for Imbalanced Magnetohydrodynamic Turbulence

C. S. Ng and T. J. Dennis

Geophysical Institute, University of Alaska Fairbanks

Abstract

We present results from a series of three-dimensional simulations of magnetohydrodynamic (MHD) turbulence based on reduced MHD equations. Alfven waves are launched from both ends of a long tube along the background uniform magnetic field so that turbulence develops due to collision between counter propagating Alfven waves in the interior region. Waves are launched randomly with specified correlation time $t_c$ such that the length of the tube, $L$, is greater than (but of the same order of) $V_A t_c$ such that turbulence can fill most of the tube. While waves at both ends are launched with equal power, turbulence generated is imbalanced in general, with normalized cross-helicity gets close to -1 at one end and 1 at the other end. This simulation setup allows easier comparison of turbulence properties with one-dimensional turbulence transport models, which have been applied rather successfully in modeling solar wind turbulence. However, direct comparison of such models with full simulations of solar wind turbulence is difficult due to much higher level of complexity involved. We will present our latest simulations at different resolutions with decreasing dissipation (resistivity and viscosity) levels and compare with model outputs from turbulence transport models and discuss the validity of different assumptions employed in such models. This work is supported by a NASA grant NNX15AU61G.
Outline

• Motivation/Introduction: simulations of solar coronal heating from the Parker regime to the turbulence heating regime.

• Turbulence heating: Kolmogorov cascade or Iroshnikov-Kraichnan (IK) cascade?

• New approach to answer this question by using 1D turbulence transport models.

• Conclusions
Coronal Heating: DC vs AC

- DC: Parker model of coronal heating (Parker 1972) --
  \[ V_A \tau_c \gg L, \quad \tau_c \sim L_\perp / v_p \]: correlation time of photospheric motions

- AC: waves, resonances, turbulence -- \[ V_A \tau_c \sim< L \]

TRACE 171 Å Fe IX/X ~1,000,000 K

Footpoint Twisting

Smooth Uniform Field

Non equilibrium with Current Sheet (Tangential Discontinuity)
Reduced Magnetohydrodynamics

- Reduced Magnetohydrodynamics (RMHD):
  - Low frequency, low beta limit of MHD, well suited for our simulations
  - Applicable in systems permeated by strong uniform magnetic field

\[ \frac{\partial \Omega}{\partial t} + [\phi, \Omega] = \frac{\partial J}{\partial z} + [A, J] + \nu \nabla^2 \Omega \]

\[ \frac{\partial A}{\partial t} + [\phi, A] = \frac{\partial \phi}{\partial z} + \eta \nabla^2 A \]

\[ \mathbf{B} = \mathbf{\hat{z}} + \mathbf{B}_\perp = \mathbf{\hat{z}} + \nabla \times \mathbf{A} \times \mathbf{\hat{z}} \quad \text{--- magnetic field,} \]

\[ \mathbf{v} = \nabla \phi \times \mathbf{\hat{z}} \quad \text{--- fluid velocity,} \]

\[ \Omega = - \nabla^2 \phi \quad \text{--- vorticity,} \]

\[ J = - \nabla^2 A \quad \text{--- current density,} \]

\[ \eta \quad \text{--- resistivity,} \quad \nu \quad \text{--- viscosity,} \]

\[ [\phi, A] = \phi_y A_x - \phi_x A_y \]
Investigate how heating rate scales with dissipation coefficients as model evolves in statistical steady state.

Key challenges in computational study:
1. Separation of scales required very high resolutions
2. Need long time integrations for good statistics

$L << V_A \tau_c$ limit: Parker model

• Formation of thin current layers.

\[ \eta = \nu = 0.00125 \]  
(128x128x32)
Lundquist number scaling of heating rate

- Average energy dissipation rate saturated in small $\eta$.

Longcope & Sudan (1994):
\[
\langle P_F \rangle \sim \nu_F \bar{B}_\perp \propto \eta^{-1/3}
\]
\[
\bar{B}_\perp \sim \left\{ l_F (N_E \tau_e)^{-1} \Delta^{-1/2} \right\}^{2/3} \eta^{-1/3}
\]

- Saturated level $\sim$ realistic heating rate of the corona
Scaling analysis in 3D

Sweet-Parker reconnection:  \( \frac{\delta}{\Delta} \sim S_{\perp}^{-1/2} \quad S_{\perp} \equiv \bar{B}_{\perp} w/\eta \)

Heating rate:  \( \bar{W} \sim \eta N \Delta L \frac{\bar{B}_{\perp}^2}{\delta} \sim \frac{\bar{B}_{\perp}^2 L L_{\perp}^2}{\tau_E} \)

If \( \tau_E < \tau_c \), no random walk:
\[
\bar{B}_{\perp} \sim B_z \frac{v_p \tau_E}{L} \sim \left[ \left( \frac{B_z v_p}{LN} \right)^2 \frac{L_{\perp}^4}{w \eta} \right]^{1/3}
\]
\[
\bar{W} \sim \left( \frac{L_{\perp}^{10} B_z^5 v_p^5}{L^2 N^2 w \eta} \right)^{1/3}
\]

If \( \tau_E > \tau_c \), random walk:
\[
\bar{B}_{\perp} \sim B_z \frac{v_p (\tau_c \tau_E)^{1/2}}{L} \sim \left[ \left( \frac{B_z v_p L_{\perp}}{L} \right)^4 \frac{\tau_c^2}{N^2 w \eta} \right]^{1/5}
\]
\[
\bar{W} \sim \frac{L_{\perp}^2}{L} B_z^2 v_p^2 \tau_c
\]

Substituting numerical parameters shows that transition at around \( \eta = 10^{-3} \)

Heating rate for $L << V_A \tau_c$

- Parker regime: $<W> = W_{PK} \sim \frac{V_A^2 v_p^2 \tau_c L_{\perp}^2}{L}$

- For our parameters ($V_A = L_{\perp} = 1$, $\tau_c = 10$, $v_p \sim <0.1$ ),

$$W_{PK} \sim \frac{v_p^2 \tau_c}{L} = \frac{10v_p^2}{L}$$

- How $<W>$ scales with $L$? Performed runs with different $L$ (keeping other parameters fixed)

- Covering the Parker regime ($V_A \tau_c >> L$) to the waves/turbulence regime ($V_A \tau_c \sim <L$)

- Runs with perpendicular resolutions of $128^2$, $256^2$ and $512^2$, and parallel resolution of $32L$ with $L$ from 1 to 24.
Heating rate for $L \rightarrow \infty$

- When $L >> L_d$, Alfvén waves launched from both ends dissipated totally due to turbulence cascade through interacting with counter propagating waves.

$$\begin{align*}
z(0) &= L - z \\
L_d &= \left( W_{turb} \sim 2v_p^2 \right) \text{ , } v_p \text{ is the boundary flow magnitude}
\end{align*}$$

- How long is the turbulence interaction length $L_d$? This should depend on whether cascade is Kolmogorov or Iroshnikov-Kraichnan (IK).
Heating rate of $W_{PK}$ and $W_{turb}$
Heating rate cf $W_{PK}$ and $W_{turb}$
Intermediate $L$ – volume filling turbulence

Current Density Concentration at $t = 0.0844700$
threshold = 0.00; $J_{\text{max}} = 8.87639$; $J_{\text{min}} = -8.49424$
Profiles integrated over $x_\perp$
Profiles integrated over $x_\perp$

$Z^2_-$

$Z^2_+$

$\sigma_c$

$J^2$

512$^2 \times 768$
Integrated over $x_\perp$ -- averaged over time

$$\sigma_c = \frac{Z_+^2 - Z_-^2}{Z_+^2 + Z_-^2}$$
Integrated over $x_\perp$ -- averaged over time

$Z^2_-$

$Z^2_+$

$\sigma_c$

$J^2$

$512^2 \times 768$
Heating rate for intermediate $L$ -- Kolmogorov

- When $L \sim < L_d$, turbulence fills the whole length.

- For Kolmogorov cascade,
  \[ \frac{dv^2}{dz} = \frac{1}{V_A} \frac{dv^2}{dt} \sim \frac{v^3}{V_A v_p \tau_c} \Rightarrow L_d \sim V_A \tau_c \]

- $<W> = W_{Kol} \sim \frac{v^3 L L_\perp^2}{v_p \tau_c} \sim 2 V_A v_p v L_\perp^2 \sim 2^{3/2} V_A v_p^2 \left( \frac{V_A \tau_c}{L} \right)^{1/2} L_\perp^2$

- For our parameters ($V_A = L_\perp = 1$, $\tau_c = 10$), $L_d \sim 10$,
  \[ <W> = W_{Kol} \sim 2^{3/2} v_p^2 \left( \frac{\tau_c}{L} \right)^{1/2} \sim 2^{3/2} v_p^2 \left( \frac{10}{L} \right)^{1/2} \]
Heating rate for intermediate $L$ -- IK

• When $L \sim < L_d$, turbulence fills the whole length.

• For Iroshnikov-Kraichnan cascade,

$$\frac{dv^2}{dz} = \frac{1}{V_A} \frac{dv^2}{dt} \sim \frac{v^4}{V_A^2 v_p \tau_c} \Rightarrow L_d \sim \frac{V_A^2 \tau_c}{v_p} >> V_A \tau_c$$

• $< W >= W_{IK} \sim \frac{v^4 LL^2_\perp}{V_A v_p \tau_c} \sim 2V_A v_p vL^2_\perp \sim 2^{4/3} V_A^{4/3} v^{5/3}_p \left(\frac{V_A \tau_c}{L}\right)^{1/3} L^2_\perp$

• For our parameters ($V_A = L_\perp = 1, \tau_c = 10$), $L_d \sim 10 / v_p$,

$$< W >= W_{IK} \sim 2^{4/3} v^{5/3}_p \left(\frac{\tau_c}{L}\right)^{1/3} \sim 2^{4/3} v^{5/3}_p \left(\frac{10}{L}\right)^{1/3} > W_{Kol}$$ for most $L$
Kolmogorov or IK?

• Assumption for IK theory: in one collision, \( \frac{\Delta v}{v} \sim \frac{k_\perp v}{k_{\parallel} V_A} \ll 1 \)

so that the energy cascade rate is

\[
\varepsilon \sim \frac{v^2}{N(1/k_{\parallel} V_A)} \sim \left( \frac{k_\perp v}{k_{\parallel} V_A} \right)^2 \frac{v^2}{(1/k_{\parallel} V_A)} \sim \frac{k_{\perp}^2 v^4}{k_{\parallel} V_A}
\]

• In our case: \( k_\perp \sim \frac{1}{v_p \tau_c} \), \( k_{\parallel} \sim \frac{1}{V_A \tau_c} \), \( v > v_p \Rightarrow \frac{k_\perp v}{k_{\parallel} V_A} > \frac{v_p V_A \tau_c}{v_p \tau_c V_A} \sim 1 \)

• This means that the IK assumption is not satisfied. Or the turbulence is strong when Alfvén waves are launched by random boundary flows => Kolmogorov cascade
Heating rate of $W_{Kol}$ and $W_{IK}$
Heating rate cf all scalings

\[ W_{\text{turb}} \]
\[ W_{\text{pk}} \]
\[ W_{\text{turb}} + W_{\text{pk}} \]
\[ W_{\text{kol}} \]
\[ W_{\text{ik}} \]
Solar wind turbulence model

The steady state solar wind turbulence model developed by [Matthaeus et al. 1994, 1996] and later developments:

\[
\frac{dZ^2}{dr} = - \frac{AZ^2}{r} - \frac{\alpha Z^3}{\lambda V_{SW}} + \frac{Q}{V_{SW}}
\]

\[
\frac{d\lambda}{dr} = - \frac{C \lambda}{r} + \frac{\beta Z}{V_{SW}} - \frac{\beta \lambda Q}{V_{SW} Z^2}
\]

\[
\frac{dT}{dr} = - \frac{4T}{3r} + \frac{m \alpha Z^3}{3k_B \lambda V_{SW}}
\]

$Z^2$: average turbulence energy with $Z_+^2 = Z_-^2$

$\lambda$: turbulence correlation length

$T$: solar wind proton temperature

$Q$: turbulence generation rate due to pickup ions
Solar wind model with IK cascade

[Ng et al., J. Geophys. Res., 115, A02101 (2010)]

The solar wind turbulence model changed to [Matthaeus et al. 1994, Hossain et al. 1995]:

\[
\frac{dZ^2}{dr} = -\frac{AZ^2}{r} - \frac{\alpha Z^4}{\lambda V_{SW} V_A} + \frac{Q}{V_{SW}}
\]

\[
\frac{d\lambda}{dr} = -\frac{C\lambda}{r} + \frac{\beta}{V_{SW}} \left(\frac{Z^4}{V_A}\right)^{1/3} - \frac{\beta \lambda Q}{V_{SW} Z^2}
\]

\[
\frac{dT}{dr} = -\frac{4T}{3r} + \frac{m\alpha Z^4}{3k_B \lambda V_{SW} V_A}
\]

$Z^2$: average turbulence energy with $Z_+^2 = Z_-^2$

$\lambda$: turbulence correlation length

$T$: solar wind temperature

$Q$: turbulence generation rate due to pickup ions
Comparisons with observations

cf. [Isenberg et al. 2003]
Comparisons with observations

\[ Z^2/Z^2(1\text{AU}) \]

Kolmogorov without Q
Kolmogorov with Q
IK with Q
IK without Q

cf. [Smith et al. 2001]
Comparisons with observations

cf. [Smith et al. 2001]
Comparisons with observations

(cf. [Smith et al. 2006])

(a) $T_p$ vs $R$ (AU)

Kolmogorov with $Q$

(b) $T_p$ vs $R$ (AU)

IK with $Q$

- observations
- theory
Turbulence transport models for our simulations

\[
\frac{dZ^2}{dz} = \pm \alpha \frac{Z^2 Z_\pm}{V_A \lambda_\pm} \quad \text{for Kolmogorov}
\]

\[
\frac{dZ^2}{dz} = \pm \alpha \frac{Z^2 Z^2_\pm}{V_A^2 \lambda_\pm} \quad \text{for IK}
\]

\[
\frac{d\lambda_\pm}{dz} = ?
\]

[Hossain et al. 1995; Breech et al. 2005, 2008]

- Compare numerically measured length scales

\[
\left\langle \int d^2x_\perp z^2 \pm / \int d^2x_\perp \nabla^2 z^2 \pm \right\rangle^{1/2} \quad \text{and} \quad \pm \alpha \frac{Z^2 Z_\pm}{V_A} / \frac{dZ^2}{dz} \quad \text{or} \quad \pm \alpha \frac{Z^2 Z^2_\pm}{V_A^2} / \frac{dZ^2}{dz}
\]
Length-scale comparison -- Kolmogorov

\[ \frac{Z^2 Z_+}{V_A} \left/ \frac{dZ_-}{dz} \right. \]

\[ \frac{1}{\alpha} \left( \frac{\int d^2 x_{\perp} z_+^2}{\int d^2 x_{\perp} \nabla_{\perp}^2 z_-^2} \right)^{1/2} \]

\[ \alpha = 0.198 \]

512^2 \times 768
Length-scale comparison -- Kolmogorov

\[
\frac{Z^2_+ Z_-}{V_A} \left/ \frac{dZ^2_+}{dz} \right.
\]

\[
\frac{1}{\alpha} \left( \frac{\int d^2 x_\perp z^2_+}{\int d^2 x_\perp \nabla_\perp^2 z^2_+} \right)^{1/2}
\]

\( \alpha = 0.198 \)

512 \times 768
Length-scale comparison -- IK

\[-\frac{Z^2 Z_-^2}{V_A^2} \left/ \frac{dZ_-^2}{dz} \right.\]

\[\frac{1}{\alpha} \left( \frac{1}{2} \frac{\int d^2 x_\perp z_-^2}{\int d^2 x_\perp \nabla_{\perp}^2 z_-^2} \right)^{1/2}\]

\[\alpha = 10\]

512^2 \times 768
Length-scale comparison -- Kolmogorov

\[
\alpha = 10
\]
$Z$ comparison -- Kolmogorov

$Z^2$

$\alpha = 0.198$

$512^2 \times 768$
$Z_+^2$ comparison -- Kolmogorov

$\alpha = 0.198$

$512^2 \times 768$
$Z_\alpha$ comparison -- IK

$\alpha = 10$

$512^2 \times 768$
$Z^2_+$ comparison -- IK

$Z^2_+$

\[ \alpha = 10 \]

$512^2 \times 768$
Length scale equations

\[
\frac{dZ^2}{dz} = \pm \alpha \frac{Z^2 Z_\pm}{V_A \lambda_\pm}
\]

for Kolmogorov

\[
\frac{d\lambda_\pm}{dz} = \mp \frac{\beta}{V_A} Z_\pm \quad ? \quad \text{sign of } \beta?
\]

[Hossain et al. 1995]

• good fit: \[
\frac{d\lambda_\pm}{dz} = \pm \frac{\beta \sigma_c^2}{V_A} \left( Z_\pm - \frac{Z_\mp}{\alpha_1} \right), \quad \beta = -9.5, \quad \alpha_1 = 1.5, \quad \alpha = 1
\]
Model with length scale equations

\[ Z^2 \]

\[ \alpha = 1 \]

\[ \alpha_1 = 1.5 \]

\[ \beta = -9.5 \]

512^2 \times 768
Model with length scale equations

\[ Z_+^2 \]

\[ \alpha = 1 \]
\[ \alpha_1 = 1.5 \]
\[ \beta = -9.5 \]

512^2 \times 768
Model with length scale equations

\[ \sigma_c \]

\[ \beta = -9.5 \]

\[ \alpha = 1 \]

\[ \alpha_1 = 1.5 \]

512^2 \times 768
Conclusions

♦ 3D RMHD simulations for the heating of the corona have been performed for long coronal loop length $L$ in the turbulence heating regime.

♦ Time averaged profiles of quantities integrated over $x_\perp$ can be compared with 1D transport models for imbalanced turbulence (nonzero cross helicity).

♦ Simulation results are more consistent with models with Kolmogorov cascade rather than the IK cascade.

♦ Empirical equations for length scales in the 1D transport model with Kolmogorov cascade can be found to fit simulation results with reasonable agreement. The forms of these equations are somewhat different from what were used before, especially the sign of the coefficient.

♦ More simulations with higher resolutions and longer running time are needed to obtain a definite conclusion.
Runs table

**Fixed Parameters:**

<table>
<thead>
<tr>
<th></th>
<th>$\tau_E$</th>
<th>$\tau_C$</th>
<th>$k_c$</th>
<th>$\eta$</th>
<th>$\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>9.5</td>
<td>10.0</td>
<td>12.0</td>
<td>0.0003125</td>
<td>0.0003125</td>
</tr>
</tbody>
</table>

**Low-Cadence Runs:**

<table>
<thead>
<tr>
<th>Class</th>
<th>$L$</th>
<th>Resolution</th>
<th>$&lt; W &gt;$</th>
<th>$&lt; I &gt;$</th>
<th>$&lt; B_{\perp} &gt;$</th>
<th>$&lt; v_p &gt;$</th>
<th>$T_{\text{final}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>1.0</td>
<td>$128^2 \times 32$</td>
<td>0.06938</td>
<td>0.06128</td>
<td>0.57910</td>
<td>0.07350</td>
<td>0.34595E+04</td>
</tr>
<tr>
<td>S2</td>
<td>2.0</td>
<td>$128^2 \times 64$</td>
<td>0.04111</td>
<td>0.03994</td>
<td>0.45371</td>
<td>0.07451</td>
<td>0.70844E+04</td>
</tr>
<tr>
<td>S3</td>
<td>4.0</td>
<td>$128^2 \times 128$</td>
<td>0.02477</td>
<td>0.02465</td>
<td>0.36415</td>
<td>0.07431</td>
<td>1.02630E+04</td>
</tr>
<tr>
<td>S4</td>
<td>8.0</td>
<td>$128^2 \times 256$</td>
<td>0.01747</td>
<td>0.01747</td>
<td>0.32099</td>
<td>0.07315</td>
<td>1.04704E+04</td>
</tr>
<tr>
<td>S5</td>
<td>12.0</td>
<td>$128^2 \times 384$</td>
<td>0.01557</td>
<td>0.01560</td>
<td>0.31986</td>
<td>0.07326</td>
<td>1.02400E+04</td>
</tr>
<tr>
<td>S6</td>
<td>16.0</td>
<td>$128^2 \times 512$</td>
<td>0.01422</td>
<td>0.01425</td>
<td>0.31616</td>
<td>0.07370</td>
<td>1.02400E+04</td>
</tr>
<tr>
<td>S7</td>
<td>24.0</td>
<td>$128^2 \times 768$</td>
<td>0.01310</td>
<td>0.01315</td>
<td>0.32305</td>
<td>0.07515</td>
<td>1.02400E+04</td>
</tr>
</tbody>
</table>

**High-Cadence Runs:**

<table>
<thead>
<tr>
<th>Class</th>
<th>$L$</th>
<th>Resolution</th>
<th>Driving</th>
<th>$T_{\text{final}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>HC1</td>
<td>16.0</td>
<td>$128^2 \times 512$</td>
<td>$L = 0$</td>
<td>0.40942E+02</td>
</tr>
<tr>
<td>HC2</td>
<td>16.0</td>
<td>$128^2 \times 512$</td>
<td>$L = 0$, $L = 16.0$</td>
<td>0.40942E+02</td>
</tr>
</tbody>
</table>
Summary of heating rate scalings

\[ W_{PK} \sim \nu_p^2 \frac{\tau_c}{L} \]
\[ W_{turb} \sim 2\nu_p^2 \]
\[ W_{Kol} \sim 2^{3/2} \nu_p^2 \left( \frac{\tau_c}{L} \right)^{1/2} \]
\[ W_{IK} \sim 2^{4/3} \nu_p^{5/3} \left( \frac{\tau_c}{L} \right)^{1/3} \]

- Note that \( W_{PK} = W_{turb} = W_{Kol} / 2 \) at \( L \equiv L_c = \frac{\tau_c}{2} = \frac{V_A \tau_c}{2} \), the cross-over length between the Parker and the turbulence regimes
Unifying Parker and turbulence heating

• Note that for $L$ close to $L_c$, or $\left|1 - \frac{\tau_c}{2L}\right| \ll 1$

$$W_{Kol} \sim 2^{3/2} v_p^2 \left(\frac{\tau_c}{L}\right)^{1/2} = 2^{3/2} v_p^2 \left[2 - \left(2 - \frac{\tau_c}{L}\right)\right]^{1/2}$$

$$\approx 4v_p^2 \left[1 - \frac{1}{2} \left(1 - \frac{\tau_c}{2L}\right)\right] = v_p^2 \left[2 + \frac{\tau_c}{L}\right]$$

$$\sim W_{turb} + W_{PK}$$

• For $L \ll V_A \tau_c = L_d$, $W_{PK} \gg W_{turb}$; for $L \gg V_A \tau_c$, $W_{turb} \gg W_{PK}$;

  for $L \sim L_c = L_d / 2$, $W_{Kol} \sim W_{turb} + W_{PK}$

• Therefore, $W_{turb} + W_{PK}$ is a good approximation for the heating rate over all $L$. 