Bernstein-Greene-Kruskal (BGK) Modes in Two or Three Dimensions

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Journal Club, November 14, 2008
Outline

What is a BGK mode?

1-D BGK mode

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Conclusion
What is a BGK mode?

An exact nonlinear solution of the Vlasov-Poisson system of equations
What is a BGK mode?

**Vlasov equation**

\[ \frac{df_s}{dt} = 0 \]

Collisionless

EM forces
What is a BGK mode?

Vlasov equation

\[ \frac{df_s}{dt} = \frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} + \frac{q_s}{m_s} \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0 \]
What is a BGK mode?

Vlasov equation

\[ \frac{df_s}{dt} = \frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} + \frac{q_s}{m_s} \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0 \]

Special solution: \( f(C_1, C_2, \ldots) \)

\( C_1, C_1, \ldots \) are constants of motion
What is a BGK mode?

Vlasov equation -- electrostatic

\[ \frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} - \frac{q_s}{m_s} \nabla \psi \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0 \]

\[ E = -\nabla \psi \]
What is a BGK mode?

Vlasov-Poisson equation

\[
\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} - \frac{q_s}{m_s} \nabla \psi \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0
\]

\[
\nabla \cdot \mathbf{E} = -\nabla^2 \psi = 4\pi \rho = 4\pi \sum_s q_s \int d\mathbf{v} f_s
\]
What is a BGK mode?

Vlasov-Poisson equation -- linear solution

\[ f = f_0 + f_1 \]

\[ \nabla^2 \psi = 4\pi \int d\mathbf{v} f_1 \]

\[ \frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{r}} + \frac{e}{m_e} \nabla \psi \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0 \]

Plasma oscillation: Tonks & Langmuir (1929), Vlasov (1938), Bohm & Gross (1949)

Collisionless damping: Landau (1946):
What is a BGK mode?

Vlasov-Poisson equation -- steady state nonlinear solution

\[- \frac{\partial f}{\partial t} = \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \nabla \psi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0\]

\[\nabla^2 \psi = \int df - 1\]

Normalized; uniform ion background (for simplicity)

1-D solution: *Bernstein, Greene & Kruskal* (1957)

Traveling solution by change of reference frame:

\[f(x - v_0 t, v - v_0)\]

No Landau damping
1-D BGK mode

Construction method [BGK 1957]

\[ v \frac{\partial f(x,v)}{\partial x} + \frac{d\psi(x)}{dx} \frac{\partial f(x,v)}{\partial v} = 0 \]

\[ \frac{d^2\psi(x)}{dx^2} = \int dv f(x,v) - 1 \]

1st equation can be solved by:

\[ f = f(w) \]

\[ w = v^2/2 - \psi(x) \]

Integral-differential equation:

\[ \frac{d^2\psi(x)}{dx^2} = 2 \int_{-\psi(x)}^{\infty} \frac{dw f(w)}{\sqrt{2[w + \psi(x)]}} - 1 \]

Can be solved by given \( f(w) \) or \( \psi(x) \).
Single-humped potential

1-D BGK mode

Bi-polar electric field

No total net charge

[Chen 2002]
Electrons can either be trapped, or can pass through. Distribution functions for trapped and passing electrons can be different. To support a positive potential, trapped electrons are less dense -- electron holes.

[Chen 2002]
$\psi(x) = \psi_0 \exp(-x^2 / 2\delta^2)$

\[
f = \begin{cases} 
\frac{2\sqrt{-w}}{2\pi\delta^2} & \exp(-w) / \sqrt{2\pi}, \quad \text{for } w > 0 \\
1 - 2\ln\left(\frac{-4w}{\psi_0}\right) + \frac{\exp(-w)}{\sqrt{2\pi}} \left[1 - \text{erf}\left(\sqrt{-w}\right)\right], \quad \text{for } w < 0 
\end{cases}
\]
1-D BGK mode

Physical meaning

Electron velocity increases in the center, so electron density decreases

\[
\frac{d^2\psi(x)}{dx^2} = 2 \int_{-\psi(x)}^{\infty} \frac{d\omega f(\omega)}{\sqrt{2[\omega + \psi(x)]}} - 1
\]

\[
\frac{d^2\psi(x)}{dx^2} = 2 \int_{-\psi(x)}^{\infty} \frac{d\omega f(\omega)}{|v|} - 1
\]

since

\[
w = \frac{v^2}{2} - \psi(x)
\]
Formation and Coalescence of Electron Solitary Holes

K. Saeki, (a) P. Michelsen, H. L. Pécseli, and J. Juul Rasmussen
Association EURATOM–Risø National Laboratory, DK–4000 Roskilde, Denmark
(Received 6 September 1978)

Electron solitary holes were observed in a magnetized collisionless plasma. These holes were identified as Bernstein–Green–Kruskal equilibria, thus being purely kinetic phenomena. The electron hole does not damp even though its velocity is close to the electron thermal velocity. Two holes attract each other like particles of negative mass, and coalesce when their relative velocity is small.
BGK mode observations

Laboratory experiments

Recent experiments, e.g., [Danielson et al., 2004]

![Graph showing time pictures of a soliton and an electron hole at different distances. Applied potential, \( \Phi_a \), and measured potential \( \phi \).]
NONLINEAR EVOLUTION OF A TWO-STREAM INSTABILITY*

K. V. Roberts† and H. L. Berk
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(Received 12 June 1967)

Calculations of a two-stream instability have been made by following the motion of the phase-space boundaries of an incompressible and constant-density phase-space fluid. Because of the condensation of holes, which to a good approximation act as gravitational particles, large-scale nonlinear pulses develop.
**BGK mode observations**

**Numerical simulations**

FIG. 1. Evolution in phase space of a two-stream instability from time steps 200 to 600 at intervals of 50 steps. Each step is 1/20 of a plasma period, and the horizontal and vertical coordinates are $x$, $v$, respectively. Periodic boundaries have been imposed and three identical periods are shown along each row. The shaded area represents the $f=0$ region enclosed by the plasma fluid.
Long-Time Behavior of Nonlinear Landau Damping

Giovanni Manfredi

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(Received 9 June 1997)

The evolution of an initial perturbation in a collisionless, Maxwellian plasma is studied numerically. Accurate long-time simulations (up to 1600 inverse electron plasma frequencies) show that the electric field does not decay to zero, in disagreement with recent analytical results [M. B. Isichenko, Phys. Rev. Lett. 78, 2369 (1997)]. Instead, after some initial damping, the field amplitude starts to oscillate around an approximately constant value, and the phase-space distribution develops a vortex structure which survives throughout the simulation. [S0031-9007(97)04171-9]
**BGK mode observations**

**Numerical simulations**

![Image](image.png)

**FIG. 3.** Phase-space shaded plot of the distribution function in the resonant region ($v_{\text{phase}} \approx 3.21$) (run I). Darker regions correspond to regions of higher density. Regions where $f > 0.008$ are black.
Observations of Double Layers and Solitary Waves in the Auroral Plasma

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(Received 29 January 1982)

Small-amplitude double layers and solitary waves containing magnetic-field-aligned electric field components have been observed for the first time in the auroral plasma between altitudes of 6000 and 8000 km in association with electron and ion velocity distributions that indicate the presence of electric fields parallel to the magnetic field. The double layers may account for a large portion of the parallel potential drop that accelerates auroral particles.
FIG. 1. The two perpendicular and one parallel electric field components shown. Examples of double layers (DL), solitary waves (SW) and electrostatic ion cyclotron (EIC) are marked. These data were acquired on August 11, 1976, at an altitude of 6030 km, an invariant latitude of 74.1°, and a magnetic local time of 15.74 h.
Electrostatic Solitary Waves (ESW) in the magnetotail: BEN wave forms observed by GEOTAIL

H. Matsumoto¹, H. Kojima¹, T. Miyatake¹, Y. Omura¹, M. Okada¹, I. Nagano² and M. Tsutsui³

Abstract. Wave forms of BEN (Broadband Electrostatic Noise) in the geomagnetic tail were first detected by the Wave Form Capture receiver on the GEOTAIL spacecraft. The results show that most of the BEN in the plasma sheet boundary layer (PSBL) are not continuous broadband noise but are composed of a series of solitary pulses having a special form which we term "Electrostatic Solitary Waves (ESW)". A nonlinear BGK potential model is proposed as the generation mechanism for the ESW based upon a simple particle simulation which considers the highly nonlinear evolution of the electron beam instability. The wave forms produced by this simulation are very similar to those observed by GEOTAIL and suggest that the nonlinear dynamics of the electron beam play an essential role in the generation of ESW.
BGK mode observations

Figure 1. Example of the dynamic spectra of BEN in the Plasma Sheet Boundary Layer and its wave forms in the time domain from the WFC instrument on GEOTAIL. The white line shows the local electron cyclotron frequency.
BGK mode observations

Space-based observations


WIND observations of coherent electrostatic waves in the solar wind

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Phase-space electron holes along magnetic field lines

L. Muschietti, R. E. Ergun, I. Roth, and C. W. Carlson

Abstract. Recent observations from satellites crossing active magnetic field lines have revealed solitary potential structures that move at speeds substantially greater than the ion thermal velocity. The structures appear as positive potential pulses rapidly drifting along the magnetic field. We interpret them as BGK electron holes supported by a population of trapped and passing electrons. Using Laplace transform techniques, we analyse the behavior of one phase-space electron hole. The resulting potential shapes and electron distribution functions are self-consistent and compatible with the field and particle data associated with the observed pulses. In particular, the spatial width increases with increasing amplitude. The stability of the analytic solution is tested by means of a two-dimensional particle-in-cell simulation code with open boundaries. We also use our code to briefly investigate the influence of the ions. The nonlinear structure appears to be remarkably resilient.
BGK mode observations

Width-amplitude relation

Figure 2. Width-amplitude relation. Statistical determination from the FAST dataset vs prediction for a BGK electron hole. See text for details.
Debye-Scale Plasma Structures Associated with Magnetic-Field-Aligned Electric Fields


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(Rceived 4 March 1998)

We report a new type of spatially coherent plasma structure that is associated with quasistatic, magnetic-field-aligned electric fields in space plasmas. The solitary structures form in a magnetized plasma, are multidimensional, and are highly supersonic. The size along $B_0$ is a few $\lambda_D$ and increases with increasing amplitude, unlike a classical soliton. The perpendicular size appears to be influenced by ion motion. We show that the structures facilitate ion-electron momentum exchange and suggest that an aggregate of structures may play a role supporting large-scale, parallel electric fields.

[S0031-9007(98)06705-2]
**BGK mode observations**

Three-dimensional features

FIG. 3. Data from the Fast Auroral SnapshoT (FAST) satellite showing 3D electrostatic solitary waves observed in the auroral ionosphere. The pulses are bipolar in the parallel electric field $E_\parallel$ and are unipolar in both components of perpendicular electric field $E_\perp$ (from [Ergun et al. 1998]).
BGK mode observations

Three dimensional features


Solitary waves observed in the auroral zone: the Cluster multi-spacecraft perspective

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M. André³, A. Fazakerley¹⁰, A. Balogh¹¹, and H. Rème⁹
BGK mode observations

CLUSTER WBD WAVEFORM DATA
9.5 kHz Bandwidth

Electric Field (mV/m)

Time (Seconds from 11:17:19.46 UT on February 27, 2002)
Solution for infinite B

BGK electron solitary waves in 3D magnetized plasma

Li-Jen Chen$^{1,2}$ and George K. Parks$^{2,3}$

[1] This paper presents analytical solutions that we obtained in extending the BGK electron solitary wave solutions in 1D to include the 3D electrical interaction ($E \sim 1/r^2$) of charged particles. Our results indicate that for a single humped electric potential, the parallel cut of the perpendicular component of the electric field ($E_\perp$) is unipolar and that of the parallel component ($E_\parallel$) bipolar. The multi-dimensional features of solitary waves have been observed by the FAST satellite. The parallel width-amplitude relation is found to be an inequality and it depends on the perpendicular scale size of the solitary structure. This feature can be used in conjunction with experimental data to obtain an estimate on the typical perpendicular size of observed solitary waves.
**3-D BGK mode theory**

**Solution for infinite B**

**Basic assumption:** electrons moving along $\mathbf{B}$ only

$\Rightarrow$ back to 1-D problem

**Electric potential:**

$$\Phi(\mathbf{r}) = \phi_{\parallel}(z) J_0 \left( k_{00} \frac{r}{r_s} \right)$$

**Steady state Vlasov equation:**

$$v \frac{\partial F(r, z, v)}{\partial z} + \frac{1}{2} \frac{\partial \Phi(r, z)}{\partial z} \frac{\partial F(r, z, v)}{\partial v} = 0$$

**Poisson-Vlasov equation:**

$$\frac{\partial^2 \phi(z)}{\partial z^2} - k^2 \phi(z) = \int_{-\Phi}^{0} dw \frac{f_{tr}(w)}{2\sqrt{w+\phi}} + \int_0^\infty dw \frac{f_p(w)}{2\sqrt{w+\phi}} - 1,$$

[Chen & Parks 2002]
3-D BGK mode theory

Solution for infinite B

[Chen & Parks 2002]
3-D BGK mode theory

Width-amplitude relation -- inequality

\[ \psi \gg 1, \delta \geq \sqrt[4]{\psi} \]

for \( \psi \ll 1, \delta \geq \psi^{1/4} \)

[Chen & Parks 2002]

Figure 9. The potential amplitudes and parallel widths of 3D electron holes observed by Polar PWI compared with the theoretical bounding curve for \( M = 0.3, T_e/T_i = 0.1, \) and \( \delta_r = 2. \) A large fraction of data points lie significantly away from the bounding curve, presenting the evidence that Nature can realize electron holes that dwell not just on the bounding curve but deep into the allowed space. These electron holes were observed in the Polar cusp.

[Chen et al. 2005]
3-D BGK mode theory

Extension to finite $\mathbf{B}$?

Exact solutions not yet found for finite $\mathbf{B}$, although approximate solutions found using drift-kinetic equation requiring strong $\mathbf{B}$ e.g. Jovanovic & Shukla (2000) Grabbe (2005)

No solution for $\mathbf{B} = 0$?  [Chen 2002]

Only true for $f = f(w)$
3-D BGK mode for $B = 0$

No solution if $f$ only depends on $w$

1-D: \[-\rho = \frac{d^2\psi(x)}{dx^2} = 2 \int_{-\psi(x)}^{\infty} \frac{d\psi(w)}{\sqrt{2[w+\psi(x)]}} - 1 = 2 \int_{-\psi(x)}^{\infty} \frac{d\psi(w)}{|v|} - 1\]

2-D: \[-\rho = \nabla^2 \psi = 2\pi \int_{-\psi}^{\infty} d\psi(w) - 1\]

3-D: \[-\rho = \nabla^2 \psi = 4\pi \int_{-\psi}^{\infty} |v| d\psi(w) - 1\]

For 2-D and 3-D, \[\frac{d\rho}{d\psi} \leq 0\]

3-D BGK mode for $B = 0$

solution dependent on angular momentum

3-D BGK solution for $B = 0$ not constructed since [BGK 1957]!

For a spherical potential $\psi = \psi(r)$

$f = f(w,l)$ with $l = v_\perp r$ is also a solution of the Vlasov equation

Possible to support the electric field self-consistently --- satisfying the Vlasov-Poisson equation
3-D BGK mode for $B = 0$

**Toy example**

\[ f = g(r) \delta(v_r) \delta \left( v_\perp^2 + r \frac{d\psi}{dr} \right) \]

Poisson equation $\Rightarrow$

\[ \frac{1}{r} \frac{d^2(r\psi)}{dr^2} = \pi g(r) - 1 \]

for $\psi(r) = \psi_0 \exp(-r^2/\delta^2)$

requires width-amplitude inequality

$\delta^2 > 6\psi_0$

similar to Chen’s results
More general solution

\[ f(w,l) = \frac{1}{(2\pi)^{3/2}} e^{-w} f_1(l) \]

still a particular choice: \( f \to \exp(-v^2/2)/(2\pi)^{3/2} \)

One possible choice: \( f_1(v_\perp r) = 1 - (1 - h_0) \exp(-v_\perp^2 r^2/x_0^2) \)

\( f_1(0) = h_0 \geq 0 \)

Slight preference against zero angular momentum
3-D BGK mode for $B = 0$

More general solution

Poisson equation $\Rightarrow$

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} (r \psi) \right) = e^\psi h(r) - 1$$

$$h(r) = \left( h_0 + \frac{2r^2}{x_0^2} \right) / \left( 1 + \frac{2r^2}{x_0^2} \right)$$

Boundary condition:

$$\psi(r \to \infty) \to 0 \quad \psi(r = 0) = \psi_0 \quad \psi'(r = 0) = 0$$

Existence of solution: solve $\psi_0$ for given $h_0$ and $x_0$
3-D BGK mode for $B = 0$
Numerical solution (for $0 \leq h_0 < 1$)

Electron hole

$\psi(r \to \infty) \to \psi_\infty / r^2$

No total net charge

Fig. 1  (a) Numerical solution $\psi(r)$ for the case with $h_0 = 0.9$ and $x_0 = 1$. (b) The same solution in log-log plot. The dashed line showing $\psi_\infty = x_0^2 (1 - h_0) / 2$. (c) Radial electric field. (d) Normalized charge density $1 - e^{\psi} h(r)$. 
3-D BGK mode for $B = 0$
Numerical solution (for $h_0 > 1$)

Negative potential, not possible in 1-D without ion dynamics

Mathematically possible not sure how physical

Fig. 2  (a) Numerical solution $\psi(r)$ for the case with $h_0 = 1.1$ and $x_0 = 1$.  (b) The same solution in log-log plot.  The dashed line showing $\psi_\infty = x_0^2(1 - h_0) / 2$.  (c) Radial electric field.  (d) Normalized charge density $1 - e^\psi h(r)$.
Width-amplitude inequality

Fig. 3 Plots of $\psi_0$ vs. $x_0$ for various values of $h_0$, from 0.1 (top curve) to 0.9 (bottom curve) in an increment of 0.1. The dashed line is $\psi_0 = 8x_0^2(1 - h_0)$ for $h_0 = 0.1$.

**Figure 9.** The potential amplitudes and parallel widths of 3D electron holes observed by Polar PWI compared with the theoretical bounding curve for $M = 0.3$, $T_e/T_i = 0.1$, and $\delta_r = 2$. A large fraction of data points lie significantly away from the bounding curve, presenting the evidence that Nature can realize electron holes that dwell not just on the bounding curve but deep into the allowed space. These electron holes were observed in the Polar cusp.

[Chen et al. 2005]
3-D BGK mode for finite B

Equation to be solved

Cylindrically symmetric potential: \( \psi = \psi(\rho, z) \)

Vlasov solution: \( f = f(w, 2\rho v_\phi + B\rho^2) \)

Poisson eqn:

\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \psi}{\partial \rho} \right) + \frac{\partial^2 \psi}{\partial z^2} = \int d^3v f \left( \frac{v^2}{2} - \psi, 2\rho v_\phi + B\rho^2 \right) - 1
\]

More difficult to solve, but may be more relevant to observations, may even be more stable.
2D BGK mode for finite $B$

Cylindrically symmetric potential: $\psi = \psi(\rho)$

try a particular form: $f(w,l) = \left(2\pi\right)^{-3/2} \exp(-w) \left[1 - h_0 \exp(-kl^2)\right]$

where $l = 2\rho v_\phi + B\rho^2$

Poisson equation $\Rightarrow$

$$\frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{d\psi}{d\rho} \right) = e^{\psi(\rho)} \left[ 1 - \frac{h_0}{\sqrt{1 + 8k\rho^2}} \exp\left( -\frac{kB^2\rho^4}{1 + 8k\rho^2} \right) \right] - 1$$

2D BGK mode for finite $B$ -- Numerical solution

(a) $\psi(\rho)$ (b) $\psi(\rho)$ in semi-log plot. Dashed line: $\psi = \propto \exp(-\rho)$.
(c) Radial electric field.
(d) Normalized charge density.
2D BGK mode for finite B: width-amplitude inequality

[c.f. Chen et al. 2005]
Solution for Vlasov-Poisson-Ampère equations

These solutions have finite current density due to $l$ dependence.

Current density produces self-consistent $\mathbf{B}$ field:

$$\nabla \times \mathbf{B} = -\beta_e^2 \int d^3v f_0 v$$

with $\beta_e = v_e / c$

Solution for Vlasov-Poisson-Ampère equations found.

Magnetic field correction is small for $\beta_e \ll 1$, but can be significant even for $\beta_e \sim 0.1$. 
3D dust BGK mode as a dust void?

Void in a dusty plasma: no dust within a volume. [Samsonov & Goree, 1999]

Recent works on fluid theory of dust void: [Ng et al., 2007]

BGK mode as a dust void? [Jovanovic and Shukla 2003]

3D dust BGK mode [Ng & Bhattacharjee 2006]
3-D BGK mode

Future works

(i) dynamical accessibility: how can such a 3D BGK state be produced?
(ii) instability: how unstable? can it be stabilized?
(iii) collision: what happens if a weak collision is at work?
(iv) 3D exact solution with finite $B$
(v) compare with observations
Conclusion

◊ Localized BGK solution does not exist in 2D/3D unmagnetized plasmas if the distribution function only depends on energy.

◊ Solutions possible in 3D/2D if distribution function also depends on angular momentum.

◊ Examples of such 3D/2D solutions constructed.

◊ Properties of these solutions analyzed, including the width-amplitude relation that is consistent with studies of other BGK solutions [e.g., Chen 2002].

◊ Works are ongoing. We welcome comments and collaborations, especially from the observation side.
Existence of solution

For $h_0 = 1$: $f_1 = h = 1$ for all $r$, so, no non-trivial solution.

For $0 \leq h_0 < 1$:

(i) if $\psi_0 > \ln(1/ h_0)$, $\psi (r \to \infty) \to \infty$;

(ii) if $\psi_0$ small enough or negative, $\psi (r \to \infty) \to -\infty$.

(iii) must exist a $\psi_0$ in between s. t. $\psi (r \to \infty) \to 0$.

For $h_0 > 1$:

(i) if $\psi_0 < \ln(1/ h_0) < 0$, $\psi (r \to \infty) \to -\infty$;

(ii) if $|\psi_0|$ small enough or positive, $\psi (r \to \infty) \to \infty$.

(iii) must exist a $\psi_0$ in between s. t. $\psi (r \to \infty) \to 0$.

Effectively, a problem of solving for $\psi_0$. 