Electrostatic Structures in Space Plasmas: Properties of Two-dimensional Magnetic Bernstein-Greene-Kruskal Modes

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Observations of Double Layers and Solitary Waves in the Auroral Plasma

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Small-amplitude double layers and solitary waves containing magnetic-field-aligned electric field components have been observed for the first time in the auroral plasma between altitudes of 6000 and 8000 km in association with electron and ion velocity distributions that indicate the presence of electric fields parallel to the magnetic field. The double layers may account for a large portion of the parallel potential drop that accelerates auroral particles.
FIG. 1. The two perpendicular and one parallel electric field components shown. Examples of double layers (DL), solitary waves (SW) and electrostatic ion cyclotron (EIC) are marked. These data were acquired on August 11, 1976, at an altitude of 6030 km, an invariant latitude of 74.1°, and a magnetic local time of 15.74 h.
BGK mode observations in space


WIND observations of coherent electrostatic waves in the solar wind

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Electrostatic Solitary Waves (ESW) in the magnetotail: BEN wave forms observed by GEOTAIL

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Abstract. Wave forms of BEN (Broadband Electrostatic Noise) in the geomagnetic tail were first detected by the Wave Form Capture receiver on the GEOTAIL spacecraft. The results show that most of the BEN in the plasma sheet boundary layer (PSBL) are not continuous broadband noise but are composed of a series of solitary pulses having a special form which we term "Electrostatic Solitary Waves (ESW)". A nonlinear BGK potential model is proposed as the generation mechanism for the ESW based upon a simple particle simulation which considers the highly nonlinear evolution of the electron beam instability. The wave forms produced by this simulation are very similar to those observed by GEOTAIL and suggest that the nonlinear dynamics of the electron beam play an essential role in the generation of ESW.
**BGK mode observations**

![Diagram](image)

**Figure 1.** Example of the dynamic spectra of BEN in the Plasma Sheet Boundary Layer and its wave forms in the time domain from the WFC instrument on GEOTAIL. The white line shows the local electron cyclotron frequency.
**BGK mode observations**

**Width-amplitude relation**

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**Phase-space electron holes along magnetic field lines**

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**Abstract.** Recent observations from satellites crossing active magnetic field lines have revealed solitary potential structures that move at speeds substantially greater than the ion thermal velocity. The structures appear as positive potential pulses rapidly drifting along the magnetic field. We interpret them as BGK electron holes supported by a population of trapped and passing electrons. Using Laplace transform techniques, we analyse the behavior of one phase-space electron hole. The resulting potential shapes and electron distribution functions are self-consistent and compatible with the field and particle data associated with the observed pulses. In particular, the spatial width increases with increasing amplitude. The stability of the analytic solution is tested by means of a two-dimensional particle-in-cell simulation code with open boundaries. We also use our code to briefly investigate the influence of the ions. The nonlinear structure appears to be remarkably resilient.
Figure 2. Width-amplitude relation. Statistical determination from the FAST dataset vs prediction for a BGK electron hole. See text for details.
Debye-Scale Plasma Structures Associated with Magnetic-Field-Aligned Electric Fields

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We report a new type of spatially coherent plasma structure that is associated with quasistatic, magnetic-field-aligned electric fields in space plasmas. The solitary structures form in a magnetized plasma, are multidimensional, and are highly supersonic. The size along \( B_0 \) is a few \( \lambda_D \) and increases with increasing amplitude, unlike a classical soliton. The perpendicular size appears to be influenced by ion motion. We show that the structures facilitate ion-electron momentum exchange and suggest that an aggregate of structures may play a role supporting large-scale, parallel electric fields. [S0031-9007(98)06705-2]
FIG. 3. Data from the Fast Auroral SnapshoT (FAST) satellite showing 3D electrostatic solitary waves observed in the auroral ionosphere. The pulses are bipolar in the parallel electric field $E_{\parallel}$ and are unipolar in both components of perpendicular electric field $E_{\perp}$ (from [Ergun et al. 1998]).
Introduction

◇ Bernstein-Greene-Kruskal (BGK) mode [Bernstein, Greene and Kruskal, 1957] was found as exact nonlinear solution of the Vlasov-Poisson equations for a 1D plasma.

◇ 3D BGK mode with an infinitely strong background magnetic field has been found [Chen 2002].

◇ 3D BGK mode in an unmagnetized plasma can exist if it also depends on angular momentum [Ng and Bhattacharjee 2005].

◇ 2D BGK mode in a finite magnetic field was also found [Ng, Bhattacharjee, and Skiff 2006].

◇ In this work, 2D BGK mode, which is an exact solution for the Vlasov-Poisson-Ampère system, in a finite magnetic field is constructed using a more direct method. Particle-in-Cell (PIC) simulations are performed to study the stability of the BGK mode.
Vlasov equation

\[ \frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} + \frac{q_s}{m_s} \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0, \]

i.e., the Boltzmann’s equation without collision. Couple with the Maxwell’s equations:

\[ \frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E}, \quad \nabla \cdot \mathbf{B} = 0, \]

\[ \nabla \cdot \mathbf{E} = 4\pi \rho = 4\pi \sum_s q_s \int d\mathbf{v} f_s , \]

\[ \nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c} \mathbf{J} = \frac{4\pi}{c} \sum_s q_s \int d\mathbf{v} \mathbf{v} f_s . \]
Exact static solution without background $B$

$$\mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} - \frac{q_s}{m_s} \nabla \psi \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0,$$

$$\nabla^2 \psi = -4\pi \sum_s q_s \int d\mathbf{v} f_s,$$

$$\mathbf{E} = -\nabla \psi.$$

A BGK mode is an exact nonlinear solution of these equations.
Electron dynamics with uniform ion background

\[ \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \nabla \psi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 , \quad \text{--- Vlasov equation} \]

\[ \nabla^2 \psi = \int d\mathbf{v} f - 1 . \quad \text{--- Poisson equation} \]

Normalization procedure: \( \mathbf{v} \rightarrow v_e \mathbf{v} , \mathbf{r} \rightarrow \lambda \mathbf{r} , \psi \rightarrow 4\pi n_0 e \lambda^2 \psi , \) and \( f_e \rightarrow n_0 f / v_e^3 . \)

\( v_e : \) electron thermal velocity

\( \lambda = v_e / \omega_{pe} : \) electron Debye length

Note that the Vlasov equation is satisfied if \( f \) depends only on conserved quantities (energy, angular momentum, etc).
Solutions with only energy dependence

The form

\[ f = f(w) , \]

where \( w = \frac{v^2}{2} - \psi(r) \) is the energy, solves the Vlasov equation exactly.

Has to solve the Poisson equation self-consistently also.
One dimensional problem

\[ \nu \frac{\partial f(x, \nu)}{\partial x} + \frac{d\psi(x)}{dx} \frac{\partial f(x, \nu)}{\partial \nu} = 0 , \]

\[ \frac{d^2 \psi(x)}{dx^2} = \int d\nu f(x, \nu) - 1 . \]

Exact solutions of the form \( f = f(w) \) satisfying the Vlasov-Poisson simultaneously, known as the BGK modes, where first identified in [Bernstein, Greene, and Kruskal 1957]. Standard procedure of constructing these solutions can be found in many textbooks.
Example with a localized electric potential [Chen 2002]:

\[ \psi(x) = \psi_0 \exp(-x^2 / 2\delta^2) , \]

\[ f = \begin{cases} 
\frac{2\sqrt{-w/2}}{\pi\delta^2} \left[ 1 - 2\ln\left(\frac{-4w}{\psi_0}\right) \right] + \frac{\exp(-w)}{\sqrt{2\pi}} \left[ 1 - \text{erf}\left(\sqrt{-w}\right) \right], & \text{for } w < 0 \\
\exp(-w)/\sqrt{2\pi}, & \text{for } w > 0 
\end{cases} \]
Three-dimensional problem [Chen 2002]

(i) Case with a very strong background magnetic field ($B \to \infty$): effectively reduces to 1D problem with electron moving straightly along the background $B$ field.

\[
\psi(\rho, z) = \psi_0 \exp(-\rho^2 / 2\delta_\rho^2 - z^2 / 2\delta_z^2),
\]

\[
f = \begin{cases} 
\frac{2\sqrt{-w_\parallel/2}}{\pi\delta_z^2} \left[ 1 - 2\ln\left(\frac{-4w_\parallel}{\psi_0}\right) \right] - \frac{4\sqrt{-w_\parallel/2}}{\pi\delta_\rho^2} \frac{\exp(-w_\parallel)}{\sqrt{2\pi}} \left[ 1 - \text{erf}\left(\sqrt{-w_\parallel}\right) \right] & \text{for } w_\parallel > 0 \\
+ \frac{2\rho^2\sqrt{-w_\parallel/2}}{\pi\delta_\rho^2} \left( \frac{1}{\delta_\rho^2} - \frac{1}{\delta_z^2} \right) & \text{for } w_\parallel < 0
\end{cases}
\]

\[
w_\parallel = \frac{v_z^2}{2} - \psi(\rho, z)
\]
No 2D/3D localized BGK solution with $f = f(w)$ [Ng and Bhattacharjee 2005]

For 2D, $n \equiv \nabla^2 \psi = 2\pi \int_{-\psi}^{\infty} df(w) - 1$, \[ \frac{dn}{d\psi} = 2\pi f(-\psi) \geq 0. \]

For 3D, \[ g\left(\frac{v_z^2}{2} - \psi\right) = 2\pi \int_0^{\infty} v_\perp dv_\perp f\left(\frac{v_\perp^2}{2} + \frac{v_z^2}{2} - \psi\right), \]

\[ \frac{dn}{d\psi} = -\int_{-\infty}^{\infty} dv_z g'\left(\frac{v_z^2}{2} - \psi\right) \geq 0. \]

Subject to boundary condition that $\psi \to 0$ and $n \to 0$ as $|r| \to \infty$ for a localized solution.

Therefore there cannot be a point in space that $\psi$ is a local maximum (minimum) while it is positive (negative). Thus, it is impossible to construct a nontrivial solution under these restrictions.
Exact 2D solution for the Vlasov-Poisson-Ampère system

Steady-state Vlasov equation with finite nonuniform B-field:

\[
v \rho \frac{\partial f}{\partial \rho} + \left( \frac{d\psi}{d\rho} + \frac{v_{\phi}^2}{\rho} - B_z v_{\phi} \right) \frac{\partial f}{\partial v_\rho} - \left( \frac{v_\rho v_{\phi}}{\rho} - B_z v_\rho \right) \frac{\partial f}{\partial v_{\phi}} = 0
\]

Electric field: \( \mathbf{E} = -\nabla \psi(\rho) = -\frac{d\psi}{d\rho} \hat{\rho} \)

Magnetic field:

\[
\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \left[ A_\phi (\rho) \hat{\phi} \right] = \frac{1}{\rho} \frac{d(\rho A_\phi)}{d\rho} \hat{z}
\]

\[
B_z = \frac{1}{\rho} \frac{d(\rho A_\phi)}{d\rho}
\]
General Form of Exact Solutions

\[ f = f(\rho, v_\rho, v_\phi, v_z) = f(w, l) = f \left[ \frac{v_\rho^2 + v_\phi^2 + v_z^2}{2} - \psi(\rho), \rho(v_\phi - A_\phi) \right] \]

So, \( w = \frac{v_\rho^2 + v_\phi^2 + v_z^2}{2} - \psi(\rho,z) \) is proportional to energy,

\[ l = \rho(v_\phi - A_\phi) \] is proportional to angular momentum

Poisson equation:

\[ \frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{\partial \psi}{\partial \rho} \right) = \int d^3v f \left( \frac{v^2}{2} - \psi, \rho(v_\phi - A_\phi) \right) - 1 \]

with \( v^2 = v_\rho^2 + v_\phi^2 + v_z^2 \)
Ampère Law

\[
\frac{d}{d\rho} \left[ \frac{1}{\rho} \frac{d(\rho A_\phi)}{d\rho} \right] = \beta_e^2 \int d^3v v_\phi f \left( \frac{v^2}{2} - \psi, \rho(v_\phi - A_\phi) \right)
\]

\[
\beta_e^2 = \left( \frac{v_e}{c} \right)^2
\]

Boundary conditions

\[
f(w,l) \xrightarrow{\rho \to \infty} \exp \left[ -\left( \frac{v_\rho^2 + v_\phi^2 + v_z^2}{2} \right) / (2\pi)^{3/2} \right]
\]

\[
\psi(\rho) \xrightarrow{\rho \to \infty} 0, \quad A_\phi(\rho) \xrightarrow{\rho \to \infty} B_0 \rho / 2, \text{ where } B_0 \text{ is a constant}
\]

\[
\psi(0) = \psi_0, \quad \frac{d\psi}{d\rho}(0) = 0, \quad \rho A_\phi(0) = A_0, \quad \frac{1}{\rho} \frac{d(\rho A_\phi)}{d\rho}(0) = B_{00}
\]

\[B_0\] is the background \([B]\), and \([B_{00}]\) is \([B]\) at the center of the solution.
One Particular Solution

\[ f(w, l) = \frac{1}{(2\pi)^{3/2}} e^{-w} \left( 1 - h e^{-kl^2} \right) \]

After integrations, we need to solve:

\[ \frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{d\psi}{d\rho} \right) = e^{\psi(\rho)} \left[ 1 - \frac{h}{\sqrt{1 + 2k\rho^2}} \exp \left( -\frac{k\rho^2 A^2_{\phi}}{1 + 2k\rho^2} \right) \right] - 1 \]

\[ \frac{d}{d\rho} \left[ \frac{1}{\rho} \frac{d}{d\rho} (\rho A_{\phi}) \right] = -\frac{2\beta_e^2 h k \rho^2 A_{\phi}}{(1 + 2k\rho^2)^{3/2}} \exp \left( \psi - \frac{k\rho^2 A^2_{\phi}}{1 + 2k\rho^2} \right) \]

\( \psi_0 \) is determined such that when these two equations are solved together, boundary conditions are satisfied.

Then, \( B_0 \) is determined by the asymptotic behavior of \( A_{\phi} \).
Solution with very small $\beta_e$

$B_0 = B_{00}$, i.e. uniform $B$ field

(a) $\psi(\rho)$  (b) $\psi(\rho)$ in semi-log plot. Dashed line: $\psi = \alpha \exp(-\rho)$. (c) Radial electric field. (d) Normalized charge density.
Solution with larger $\beta_e$

$\beta_e^2 = 0.001$, $h = 0.9$, $k = 0.00025$

A paramagnetic localized structure.
2D BGK mode for finite B: width-amplitude inequality

[c.f. Chen et al. 2005]
PIC simulations using OOPIC Pro™

Input an exact solution as the initial condition to study stability. More difficult to input $x$ and $v$ for each electron.

Input moments of the solution instead:

$$n_e = e^{\psi(\rho)} \left[ 1 - \frac{h}{\sqrt{1 + 2k\rho^2}} \exp \left( - \frac{k\rho^2 A^2_{\phi}}{1 + 2k\rho^2} \right) \right]$$

$$\langle v \rangle = -\frac{\phi}{n_e} \frac{2hk\rho^2 A_{\phi}}{(1 + 2k\rho^2)^{3/2}} \exp \left( \psi - \frac{k\rho^2 A^2_{\phi}}{1 + 2k\rho^2} \right)$$

$$T_e = \frac{T_{e0}}{3} \left( 2 + \frac{e^{\psi}}{n_e} \left[ 1 - h \left( \frac{1 + 2k\rho^2 + (2kA_{\phi}\rho^2)^2}{(1 + 2k\rho^2)^{5/2}} \right) \exp \left( - \frac{k\rho^2 A^2_{\phi}}{1 + 2k\rho^2} \right) \right] \langle v \rangle^2 \right)$$

$\omega_{ce} = 10 \omega_{pe}$

See http://www.gi.alaska.edu/~chungsangng/bgk/bgk_b10_xyphs.mov
Moments of a 2D BGK mode -- density

\[ n_e = e^{\psi(\rho)} \left[ 1 - \frac{h}{\sqrt{1 + 2k\rho^2}} \exp\left( -\frac{k\rho^2 A_\phi^2}{1 + 2k\rho^2} \right) \right] \]
Moments of a 2D BGK mode – average velocity

\[
\langle v \rangle = -\frac{\hat{\phi}}{n_e} \frac{2h k \rho^2 A_\phi}{\left(1 + 2k \rho^2\right)^{3/2}} \exp\left(\psi - \frac{k \rho^2 A_\phi^2}{1 + 2k \rho^2}\right)
\]
Moments of a 2D BGK mode -- Temperature

\[ T_e = \frac{T_{e0}}{3} \left( 2 + \frac{e^\psi}{n_e} \right) \left[ 1 - h \left( \frac{1 + 2k\rho^2 + (2kA_\phi\rho^2)^2}{(1 + 2k\rho^2)^{5/2}} \right) \exp \left[ -\frac{k\rho^2A_\phi^2}{1 + 2k\rho^2} \right] \right] - \langle v \rangle^2 \]
Stability of BGK modes

(strong magnetic field: $\omega_{ce} = 10 \omega_{pe}$)

2D BGK mode (stable with slight modulation in frequency $\sim \omega_{ce}$)

non 2D BGK mode (still stable but with strong modulation in frequency $\sim \omega_{ce}$)

http://www.gi.alaska.edu/~chungsangng/bgk/psi61_B10_E0_-v-110311_Number_density_for_electrons.mov

http://www.gi.alaska.edu/~chungsangng/bgk/psi61_B10_E0_+v-110311_Number_density_for_electrons.mov
Stability of BGK modes

(strong magnetic field: $\omega_{ce} = 10 \omega_{pe}$)

2D BGK mode (stable with slight modulation in frequency $\sim \omega_{ce}$)

http://www.gi.alaska.edu/~chungsangng/bgk/psi61_B10_E0_-v-110311_Ey.mov

non 2D BGK mode (still stable but with strong modulation in frequency $\sim \omega_{ce}$)

http://www.gi.alaska.edu/~chungsangng/bgk/psi61_B10_E0_+v-110311_Ey.mov
Stability of BGK modes (moderate magnetic field: $\omega_{ce} = \omega_{pe}$)

2D BGK mode (stable but with strong modulation in frequency $\sim \omega_{ce}$)

http://www.gi.alaska.edu/~chungsangng/bgk/psi60_E0_drift-B0-1-110309_Number_density_for_electrons.mov

non 2D BGK mode (strong modulation in frequency $\sim \omega_{ce}$ with occasional density increase)

http://www.gi.alaska.edu/~chungsangng/bgk/psi60_B1_E0_+v_Number_density_for_electrons.mov
Stability of BGK modes (weak magnetic field: $\omega_{ce} = 0.2 \omega_{pe}$)

The original 2D BGK mode is largely filled back up with electrons but still keeps some density depletion and with fluctuations not apparently related to $\omega_{ce}$.

http://www.gi.alaska.edu/~chungsangng/bgk/psi60_B10_E0_drift-B-0.2-test1_Number_density_for_electrons.mov
One consideration regarding the stability of a 2D BGK mode is that for larger magnetic field, it actually has less total energy than that of a Maxwellian plasma.
Conclusion

◊ Exact BGK modes in 2D plasmas satisfying the Vlasov-Poisson-Ampère system are found.

◊ New solution method developed for larger electron thermal velocity using direct integration.

◊ Stability of such solution using PIC simulations is studied by using moments of the solution as input initial condition. Preliminary results show that an approximate BGK mode initial condition appears to be stable with oscillations in $\omega_{ce}$ time-scale, over many plasma times ($2\pi/\omega_{pe}$) for larger magnetic field.

◊ For weaker magnetic field, the electron density depletion (electron hole) appears to be mostly filled up but still has some reduced density with fluctuations not apparently related to $\omega_{ce}$.

◊ Initial conditions with a non BGK solution (same density hole but wrong average velocity) still keeps some density depletion, especially with stronger magnetic field, but with much larger modulations in $\omega_{ce}$ time-scale.