The Single-Zero Couplet

The simplest digital filter is the couplet \( Z - Z_0 \) with a zero located at \( Z_0 \). A quick look evaluation of this couplet's frequency behavior is readily afforded by thinking of evaluating the vector

\[ H(Z) = Z - Z_0 \]

in the complex plane with \( Z \) on the unit circle to give \( H(\omega) \). Figure 4.1 shows the vector relationships: the vector \( Z - Z_0 \) extends from the location of the zero to the unit circle. As \( Z \) moves clockwise around the unit circle from \( \omega = -\pi \) through \( \omega = 0 \) to \( \omega = \pi \), we can qualitatively sketch the magnitude and phase behavior of \( H(\omega) \); it is just the magnitude and phase of the vector \( Z - Z_0 \). Figure 4.2(a) shows the details of the phase angle of \( Z \) itself. Because its phase angle is a negative function of \( \omega \), that is \( \phi = -\omega \), it is sometimes convenient to introduce the opposite sign for phase angles, called the phase-lag angle; it is obtained by reversing the sign of all angles as shown in Fig. 4.2(b). For any operator, its phase-lag spectrum is the negative of its phase spectrum, resulting in another occupational hazard. Thus, in carefully computing the sign of the phase, we must also take care to state which angle is under consideration, the phase or phase-lag angle.
The Single-Zero Couplet

zero located inside the unit circle, always has its phase augmented by $2\pi$ as $\omega$ goes around the unit circle. In contrast, the minimum phase couplet, with its zero outside the unit circle, has its phase return to its initial value when $\omega$ goes around the unit circle. That is, for the minimum phase couplet, $\phi(\omega) = \phi(\omega + 2\pi)$; its phase has the minimum excursion in the Nyquist interval.

In addition to these phase characteristics, Fig. 4.1 allows us to make several useful observations on the frequency behavior of the couplet in terms of the zero location. First, we observe that $H(\omega)$ is, of course, periodic with period $2\pi$. For the single couplet, $H(\omega)$ is necessarily complex. As the zero becomes located closer to the unit circle, the magnitude response will drop closer to zero at the frequency that places $Z - Z_0$ closest to the zero. If the zero is located right on the unit circle, $H(\omega) = 0$ at that point. Another special location for the zero is at the origin. If the zero is located there, then clearly the magnitude spectrum is constant in $\omega$; this is called an allpass filter. The only function of an allpass filter is to delay the output. The phase varies linearly from $+\pi$ to $-\pi$ as $\omega$ goes from $-\pi$ to $\pi$. The filter becomes just $Z_0$, our familiar unit delay operator. The foregoing readily gives us a qualitative insight into the frequency behavior of the single-zero couplet in terms of the location of the zero relative to the unit circle. In addition, it is simple enough to compute the spectrum of the single-zero couplet directly.

For this purpose, it is convenient to write the couplet as

$$H(Z) = a_1 + a_2 Z,$$

which differs from our previous form only by an overall scale factor. Evaluating $H$ on the unit circle to get $H(\omega)$ gives

$$H(\omega) = a_1 + a_2 e^{-i\omega}$$

since we equate $Z = e^{-i\omega}$ there. Computing the magnitude spectrum gives

$$|H(\omega)|^2 = (a_1 + a_2 e^{-i\omega})(a_1^* + a_2^* e^{i\omega})$$

or

$$|H(\omega)| = \sqrt{a_1^2 + a_2^2 + 2 \text{Re}(a_1 a_2^* e^{i\omega})}$$

(4.2)

The phase spectrum can be calculated from

$$\phi(\omega) = \tan^{-1} \left[ \frac{\text{Im} H(\omega)}{\text{Re} H(\omega)} \right]$$

(4.3)

A property of the magnitude spectrum that is of basic importance to spectral analysis is that Eq. (4.2) is invariant under a complex conjugate
interchange of \( a_1 \) and \( a_2 \). That is to say, the two couplets

\[ H_1 = a_1 + a_2 Z \]

and

\[ H_2 = a_1^* + a_2^* Z \]

have exactly the same magnitude spectrum according to Eq. (4.2). The phase spectrum is not invariant under this exchange; it takes a minimum phase couplet into a maximum phase and vice versa.

This exchange property is sufficiently important to warrant an example. Consider the two couplets:

\[ H_1(Z) = 0.5 + Z, \quad \text{maximum phase} \]

and

\[ H_2(Z) = 1 + 0.5Z, \quad \text{minimum phase} \]

Using Eq. (4.2), we get the magnitude spectrum

\[ |H(\omega)| = \sqrt{\left(\frac{1}{2}\right)^2 + \cos \omega} \]

for both couplets. Using Eq. (4.3), we get for the two phase spectra

\[ \phi_1(\omega) = -\tan^{-1}\left(\frac{\sin \omega}{0.5 + \cos \omega}\right) \]

and

\[ \phi_2(\omega) = -\tan^{-1}\left(\frac{\sin \omega}{2 + \cos \omega}\right) \]

Figure 4.4 shows the magnitude spectra (squared) and the phase spectra of these two couplets. Many of the properties shown in these curves can be corroborated by using a sketch showing the zeros in the Z plane: the magnitude spectra are seen to be equal, at least at selected points, such as \( \omega = 0 \) and \( \omega = \pi \), and are maximum where the zeros are farthest from Z, at \( \omega = 0 \). The minimum phase couplet has the same phase value at \( \omega = -\pi \) and \( \pi \), while the phase of the maximum phase couplet increases by \( 2\pi \) over the same interval.

### The Single-Pole Couplet

The single-pole couplet is just the inverse of the single-zero couplet. As such, the same qualitative idea of the single-pole couplet’s frequency response can be obtained by thinking of the complex vector \( Z - Z_0 \) as \( Z \) moves around the unit circle. The required mental gymnastics are slightly greater than for the single-zero couplet: we must think of the inverse magnitude and the negative phase of \( Z - Z_0 \) to get the spectrum of \( 1/(Z - Z_0) \). For example, if the pole is very close to the unit circle, \(|Z - Z_0| \) becomes very small at the point on the unit circle closest to \( Z_0 \); \( H(\omega) \) becomes very large there. In general, poles close to the unit circle lead to characteristic sharp, narrow peaks in the spectra that are useful in many filter applications, such as narrow bandpass filters and sharp cutoff high-pass or lowpass filters.

To explore this fundamental property of single-pole filters, we calculate the squared magnitude spectrum. This is sometimes called the power spectrum because it is proportional to the square of the signal as is the power dissipated in an electrical load in terms of its voltage across the load or its current through the load. (The concept of the power in a digital signal will be developed further in Chapters 6 and 12; for now we can take the term power spectrum to be a slightly more efficient way of saying squared magnitude spectrum.) Proceeding then, we get the power spectrum of \( H(Z) \) by squaring it and then evaluating it on the unit circle:

\[ |H(Z)|^2 = \frac{1}{(Z - Z_0)(Z^* - Z_0^*)} - \frac{1}{Z_0^2 + Z^2 - 2 \text{Re}(Z_0 Z^*)} \]

Using polar coordinates to specify points in the complex planes gives

\[ Z = e^{-i\omega} \quad \text{and} \quad Z_0 = e^{-i\phi} = pe^{-i\omega_0} \]

where we have used the same phase for \( Z_0 \) and \( Z \) as shown in Fig. 4.2(a).

The power spectrum becomes

\[ |H(\omega)|^2 = \frac{1}{1 + \rho^2 - 2\rho \cos(\omega - \omega_0)} \]

\[ (4.4) \]
To simplify matters, we restrict our study of this spectrum to values of \( \omega \) near \( \omega_0 \) so that we can use the first two terms in the power series expansion of \( \cos(\omega - \omega_0) \):

\[
\cos(\omega - \omega_0) = 1 - (\omega - \omega_0)^2/2 + \cdots
\]

giving

\[
|H(\omega)|^2 = \frac{1}{(1 - \rho)^2 + \rho(\omega - \omega_0)^2}
\]

Our case of interest is a pole very near the unit circle. We select a stable, causal filter by placing the pole a small distance, \( \varepsilon \), outside of the unit circle:

\[
\rho = 1 + \varepsilon
\]

Thus, the power spectrum becomes for \( \omega \) near \( \omega_0 \)

\[
|H(\omega)|^2 = \frac{1}{\varepsilon^2 + (\omega - \omega_0)^2}
\]

A sketch of this equation in Fig. 4.5 shows a narrow band centered on \( \omega_0 \) with width \( 2\varepsilon \) at the \( \frac{1}{2} \) power points; poles very near the unit circle produce sharp spectral peaks.

Narrow-band filters are used as one method of searching for signals of known frequency buried in broadband noise. The narrow passband admits the signal centered on \( \omega_0 \), while rejecting the remaining frequencies, representing noise components. Of course, the desired signal must be likewise narrow banded so that it passes through the filter undistorted. In the limit \( \varepsilon \to 0 \), the signal would contain only one frequency; it would be a pure sinusoidal wave, carrying no information except, perhaps, its presence. An example of such a narrow-band filter is the radio receiving system used in continuous-wave radiotelegraphy. The signal is an on-off sinusoidal carrier wave that scarcely occupies a bandwidth of 10 to 20 Hz.

The Single-Zero, Single-Pole, Allpass Filter

The receiver contains a similarly narrow-banded filter that greatly suppresses all noise outside of this bandwidth. In past years, this filter has been implemented from electrical analog components, but it could be (for that matter the entire receiver could be) implemented from digital components.

The phase-locked detector is another example of a narrow-banded approach to weak signal detection. In this case, an oscillator is phase-locked onto the incoming signal and the noise-free oscillator is taken as the system output. The effective bandwidth of the system can be made extremely small if the phase-locked loop is permitted a long search time at the known frequency of the signal. The equivalent situation for the digital filter has a pole extremely close to the unit circle; then the recursive filter will likewise take a long time to converge.

When the phase-locked loop has the ability to track the signal to a new frequency \( \omega_0 \), the spectrum of the narrow-band filter has changed; it is a time-dependent filter that cannot be implemented by LSI systems. Digital forms of time-dependent operators can be designed in many ways, some sophisticated, others straightforward. The simplest would change the filter coefficients every so many time steps, according to a predetermined schedule.

Both the single-zero and the single-pole filters that we have just discussed are too elemental to be of practical use. For example, both have complex outputs, a feature that clearly makes them of limited practical use. However, the outputs can be made real by simply cascading the filters with another filter with its zero, or pole, at the complex conjugate of the first. In the case of the single-pole filter, we would then generate the two-pole filter

\[
F(Z) = \frac{1}{(Z - Z_0)} \times \frac{1}{(Z - Z_0^*)} = \frac{1}{Z_0^2 + Z^2 + 2Z \Re(Z_0)}
\]

which has a spectrum similar to Fig. 4.5 but with an additional peak located at \( \omega = -\omega_0 \). A corollary to this discussion, direct from the theory of polynomials, is that any real filter (or sequence) has its poles and zeros occurring in complex conjugate pairs.

The Single-Zero, Single-Pole, Allpass Filter

Apart from adding complex conjugate poles and zeros, the next most complicated filter that can be formed from our building-block couples is the single-zero, single-pole filter. A most interesting arrangement of the zero and pole is that which produces some phase spectrum but a flat magnitude spectrum—the allpass filter.

Allpass filters can be divided into four different types; only the single-zero, single-pole filter deserves any detailed attention. The other three
types are (1) the trivial constant-filter, \( H(Z) = \text{constant} \), with no effect on the magnitude or phase spectrum; (2) the pure delay with multiple zeros at the origin, \( H(Z) = Z^n \), with a linear phase spectrum; and (3) a mathematical curiosity, called the impulse delay, which we will not discuss.

The trick in making a single-zero, single-pole, allpass filter is the very special placing of the zero relative to the pole; to exactly cancel the magnitude spectra, we simply place the zero at the pole's inverse complex-conjugate location:

\[
H(Z) = \frac{Z - 1/Z_0^*}{1 - Z/Z_0^*} \tag{4.6}
\]

We next want to show that this filter has a constant magnitude spectrum or, equivalently, a constant power spectrum. The power spectrum is \( H(\omega)H(\omega)^* \), but any sequence can be expressed in terms of \( \omega \) or in terms of \( Z \). We define the power spectrum in the \( Z \) domain to be

\[
P(Z) = H(Z)H^*(1/Z) \tag{4.7}
\]

where \( H^*(1/Z) \) means to invert \( Z \) wherever it occurs and to take the complex conjugate of all coefficients. Clearly, on the unit circle this definition becomes \( H(\omega)H^*(\omega) \) because there

\[
Z = pe^{j\phi} = e^{-i\omega} \quad \text{and} \quad 1/Z = e^{-j\phi} = e^{i\omega}.
\]

Using this formulation of Eq. (4.7), our allpass filter's power spectrum becomes

\[
H(Z)H^*(1/Z) = \left| \frac{Z - 1/Z_0^*}{1 - Z/Z_0^*} \right| \frac{1/Z - 1/Z_0^*}{1 - 1/Z Z_0^*} = 1
\]

showing that the filter has a unit power spectrum; the amplitude of any sinusoidal input will be unchanged, only its phase can be altered.

To make our allpass filter of Eq. (4.6) stable and causal, we select the pole to be outside of the unit circle. Its zero is then inside and the filter is of mixed phase. Because of the zero's location, the phase of the filter will be augmented by \( 2\pi \) as \( \omega \) goes from \(-\pi\) to \( \pi \). Furthermore, we will show that the filter's phase is monotonic in \( \omega \) so that there is a phase delay at every frequency. To discuss this monotonic behavior, we introduce the derivative of interest by defining the group delay,

\[
\tau_g = \frac{d\phi}{d\omega}
\]

The Single-Zero, Single-Pole, Allpass Filter

a concept closely related to the group velocity of a wave packet. The phase-lag, \( \theta = -\phi(\omega) \), is similarly related to the phase velocity in wave propagation.

The distinction between phase velocity and group velocity is an important aspect of dispersive wave propagation. Dispersion occurs when the group delay is a function of frequency; the shape of the waveform changes during propagation. All significant causal digital systems therefore display dispersion. The case of a purely linear phase (i.e., constant group delay) and a flat magnitude spectrum is trivially achieved in digital signal processing. It is simply a time shift of the data. In sharp contrast, such behavior is very difficult to produce in analog devices where it might be highly desirable, as in amplifiers, transducers, and recorders.

Several other aspects of dispersive propagation are worth mentioning. The individual spectral components of a wave packet travel at the phase velocity while energy in the wave package is transported at the group velocity. The phase velocity of electrical signals in some materials exceeds \( c \), the velocity of light in a vacuum, while the group velocity can never exceed \( c \). All communication, such as the transfer of data between regions of a computer or among different computers, is limited by the group velocity of the signals in their conductors. Thus, in many applications, both digital and analog, the group delay is of more significance than the phase delay.

In Problem 4.9, we discuss how to show that the group delay of our single-pole, single-zero, allpass filter is given by

\[
\tau_g = \frac{(1 - 1/Z_0 Z_0^*)}{(1 - 1/Z Z_0^*)(1 - Z/Z_0^*)} \tag{4.8}
\]

Clearly, this is a positive function of frequency; the numerator is positive because the pole is outside the unit circle and the denominator is positive because it is in the form of a power spectrum in \( Z \).

Further significance of a minimum phase filter can now be seen by cascading one with our allpass filter:

\[
F = F_{\text{min}}(Z)F_{\text{allpass}}(Z)
\]

Since the phase of \( F \) is the sum of the phase of \( F_{\text{min}} \) and \( F_{\text{allpass}} \), the group delays add also. Thus, \( F \) has a greater group delay at every frequency than does the minimum phase filter. It follows that the minimum phase filter has the least group delay possible at every frequency among the class of filters with the same magnitude spectrum. We emphasize that the same statement cannot be claimed for the phase delay; the minimum phase filter does not
necessarily have the least phase delay at every frequency among the identical magnitude spectrum class of filters. See, for example, Problem 4.12.

Of course, complicated phase delays and group delays may be formed by cascading allpass filters of the single-zero, single-pole type considered here. Taking the pole near the unit circle causes most of the delay to occur near one frequency; moving the pole farther out spreads the delay over more of the spectrum.

**Elementary Filters Classified by Their Poles and Zeros**

The gross frequency behavior of ARMA filters is controlled by the relative location of their poles and zeros, relative to one another and relative to the unit circle. We would do well, then, to study in some detail the simplest ARMA filter, our single-pole, single-zero filter. In the preceding sections, we have calculated the frequency response in a complete fashion only for the single-zero filter and for the single-pole filter; for the combined case, we have limited the discussion to the particular relative placement of the zero and pole that produced the allpass filter. We will now expand the discussion with the aid of Fig. 4.6 and Fig. 4.7 to include eight possible relative positions of one pole and one zero. Cases with the pole inside of, or on, the unit circle are not considered because they represent unstable causal filters. We start with the zero at the origin and discuss the frequency response as the zero moves out along the radial line connecting the origin and the pole.

When the zero is at the origin, the numerator is $Z$, a pure delay; the denominator produces the single-pole response computed earlier (in the section on the single-pole coupler). The strong contribution of the linear phase arising from the pure delay is apparent in Fig. 4.6(a). As the zero moves off of the origin toward the pole, its magnitude spectrum starts to cancel that of the pole as seen in Fig. 4.6(b). When the zero reaches the allpass filter position relative to the pole, as in Fig. 4.6(c), its magnitude spectrum exactly cancels the magnitude spectrum of the pole leaving only the phase spectrum. As the zero leaves the allpass position and moves close to the unit circle, the magnitude spectrum of the zero falls to near zero at $\omega_0$, as shown in Fig. 4.6(d), making a narrow-band notch filter. If the zero is right on the unit circle, the frequency response of the notch filter is exactly zero at $\omega_0$, as shown in Fig. 4.7(a). Then, when the zero moves outward just off the unit circle, as in Fig. 4.7(b), a minimum phase notch filter results with a magnitude not greatly different from the case with the zero just inside the unit circle.

These notch filters are particularly useful for rejecting narrow-band interference, such as 60 Hz line noise. A more subtle application, which
will only be fully understood in later chapters, is to use the notch filter to subtract out nonharmonic frequencies in spectral estimation problems.

Next, a special situation occurs in Fig. 4.7(c) when the zero reaches the same position on the radius as the pole; they exactly cancel leaving only the trivial constant filter.

After the zero has moved beyond the pole, it has a diminishing effect on the frequency response. When the zero is far from the pole, it contributes only weakly to the magnitude and phase spectrum of the filter. If the pole is close to the unit circle, the familiar narrow-band filter will result, except an added constant from the distant zero shifts the magnitude response upward as depicted in Fig. 4.7(d); it is called a pole-on-pedestal filter. This pole-on-pedestal filter is similar to the single-pole, narrow-band filter except for the asymptotic behavior away from \( \omega_0 \). The pole-on-pedestal has a flat response away from \( \omega_0 \) while the single-pole, narrow-band filter decays away from \( \omega_0 \). The pole-on-pedestal is more convenient for designing complicated filters by cascading multiple poles because its magnitude response has little interaction with those of the other poles away from \( \omega_0 \).

The approach taken in plotting the single-zero, single-pole filter response of Figs. 4.6 and 4.7 can be readily expanded to include any number of zeros and poles. But, as explored in Problems 4.4 and 4.7, the poles that are useful in providing sharp transitions in the spectra cause long decay transients and wild phase behavior. The problem inverse to the present one under discussion is to specify a given magnitude and phase response and ask for the time domain operator that gives that response. This is the problem of digital filter design, discussed in Chapter 8.

In this chapter, we have studied the frequency response of very simple filters that were represented by two-term sequences—couplets. In the preceding chapter, where we first introduced the concept of frequency response via the sinusoidal eigenfunctions and their associated eigenvalues, we calculated the spectral response of longer but still quite simple sequences. So far, our computational methods have been analytic. Clearly, to proceed further—to conveniently compute the spectrum of a long sequence of numbers representing a complicated filter or a large data stream—we need a digital computational scheme for evaluating the spectrum. We address that interesting and important problem in the next chapter.

Problems

4.1 Give a qualitative discussion explaining the broad passband of a single-zero filter compared to the narrow passband of a single-pole filter; for example, compare Fig. 4.3(b) with Fig. 4.5.

4.2 Design a two-pole, two-zero, 2-Hz wide notch filter to reject 60-Hz line noise from data sampled at 600 Hz.

4.3 Loran is a long-range radio navigation system that employs a pulsed narrow-band signal at 100 kHz. Design a 3-term, narrow-band recursive filter with a 100 Hz bandwidth for use in improving the signal-to-noise ratio of Loran data sampled at 1000 kHz.

4.4 The penalty for employing narrow-band filters is their long decay time. Estimate the decay time of the filter used in Problem 4.3. Write a computer program to operate on synthetic data with this filter and observe the decay of its transient response by feeding it a 100 kHz sine wave sampled at 1000 kHz.

4.5 Equation (4.7) defines the power spectrum in the complex \( Z \) plane. Calculate the power spectrum for a couplet with a zero at \( \rho_0 \exp(i\omega_0) \), display its real and imaginary parts, and show that the imaginary part is identically zero only for \( Z \) on the unit circle.

4.6 Multiple-zero, multiple-pole filters can have their phase vary rapidly through many values of \( 2\pi \) during the Nyquist interval. For example, an \( n \)-zero maximum phase filter will have its phase augmented by \( 2n\pi \) over the Nyquist interval. Most methods of computing this phase will yield the result modulo \( 2\pi \), yet we know the phase of such a polynomial is continuous. Devise an algorithm to produce this continuous phase. This is called phase unwrapping.

4.7 Write a computer program to reproduce the spectral plots in Figs. 4.6 and 4.7, using your phase unwrapping ideas from Problem 4.6. Include the ability to specify arbitrary multiple poles and multiple zeros. Study single-zero, single-pole filters with the zero and pole located at different frequencies. Study multiple-pole, multiple-zero filters including those with poles and zeros occurring in conjugate pairs.

4.8 Show that as an operator relation, operating on any function of \( Z \) on the unit circle, one can write

\[
\frac{d}{d\omega} = -iZ \frac{d}{dZ}
\]

Using the results of Problem 4.8, show that the group delay of the allpass filter of Eq. (4.6) is indeed given by Eq. (4.8). Recall that the complex logarithm is given by

\[
\ln y(Z) = \ln|y(Z)| + i\phi
\]

so that for the special case of the allpass filter,

\[
\phi = \frac{1}{i} \ln|H(Z)|
\]

since \( \ln|H(Z)| = 0 \).
4.10 Using the above definition of the complex logarithm, show that for any complex spectrum \( H(\omega) \), sampled at points \( \omega \) and \( \omega + \Delta \omega \), a reasonable approximation for the group delay is
\[
\tau_g = \frac{2}{\Delta \omega} \text{Im} \left[ \frac{H(\omega + \Delta \omega) - H(\omega)}{H(\omega + \Delta \omega) + H(\omega)} \right]
\]

4.11 Integrate \( \tau_g \) of the single-zero, single-pole allpass filter over the Nyquist interval and find the filter's average group delay when it operates on 1000 Hz data. Interpret your results.

4.12 Show that the two couplets
\[
F_1 = (Z - \sqrt{2}e^{-j\pi/4})
\]
and
\[
F_2 = \sqrt{2}(Z - e^{-j\pi/4}/\sqrt{2})
\]
have the same magnitude spectrum, that \( F_1 \) is minimum phase, and that \( F_2 \) is maximum phase. Plot the zeros in the \( Z \) plane and sketch the phase lags to show that the minimum phase couplet does not have less phase delay at all frequencies.

4.13 What is the nature of the pole and zero locations of a causal digital filter?

4.14 Discuss the inverse of the allpass filter.

4.15 Show that \( \ln(Z) = \ln|y(Z)| + i\phi(Z) \).

The Discrete Fourier Transform

We have spent the last two chapters studying the frequency response of various operators. In each case, we performed an analytic calculation that resulted in a closed-form expression for \( H(\omega) \). Traditionally, closed-form solutions—for any mathematical problem—have been prized for their intrinsic elegance and beauty. Indeed, they are a satisfying outcome for any problem. But frequently in using such solutions, we like to make graphs. In making a graph, we evaluate some function at selected points, either by hand or with the aid of a computer. If a computer is used, commonly we would evaluate the function at equally spaced points sufficiently close together to give a complete picture of the solution.

In this chapter, we apply this idea of evaluating, or sampling, to the frequency response of LSI systems and operators. This frequency sampling of \( H(\omega) \) will lead us to the discrete Fourier transform, a powerful mathematical tool in digital signal processing.

Sampling the System Response in the Frequency Domain

Because sinusoids are eigenfunctions of discrete LSI systems, we have found their eigenvalues, the spectrum, extremely useful in thinking about the behavior of digital operators. In Chapter 3, we showed that the spectral response of any LSI operator can be computed from its impulse response function, \( h_\kappa \), from

\[
H(\omega) = \sum_{\kappa=0}^{N-1} h_\kappa e^{-j\omega \kappa}
\] (5.1)